Chapter 4

SPECIAL CASES IN APPLYING SIMPLEX METHODS

4.1 No Feasible Solutions

In terms of the methods of artificial variable techniques, the solution at optimality could include one or more artificial variables at a positive level (i.e. as a non-zero basic variable). In such a case the corresponding constraint is violated and the artificial variable cannot be driven out of the basis. The feasible region is thus empty.

Example 4.1. Consider the following linear programming problem.

$$\begin{array}{ll} \max & x_0 = 2x_1 + x_2 \\ \text{subject to} & \begin{cases} -x_1 + x_2 \geq 2 \\ x_1 + x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{cases} \end{array}$$

Using a surplus variable x_3, an artificial variable x_4 and a slack variable x_5, the augmented system is:

$$\begin{aligned} - x_1 + x_2 - x_3 + x_4 \quad &= 2 \\ x_1 + x_2 \quad + x_5 &= 1 \\ x_0 - 2x_1 - x_2 \quad + Mx_4 \quad &= 0 \end{aligned}$$

Now the columns corresponding to x_4 and x_5 form an identity matrix. In tableau form, we have

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	-1	1	-1	1	0	2
x_5	1	1	0	0	1	1
x_0	-2	-1	0	M	0	0

After elimination of the M in the x_4 column, we have the initial tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	-1	1	-1	1	0	2
x_5	1	1^*	0	0	1	1
x_0	$-2+M$	$-1-M$	M	0	0	$-2M$

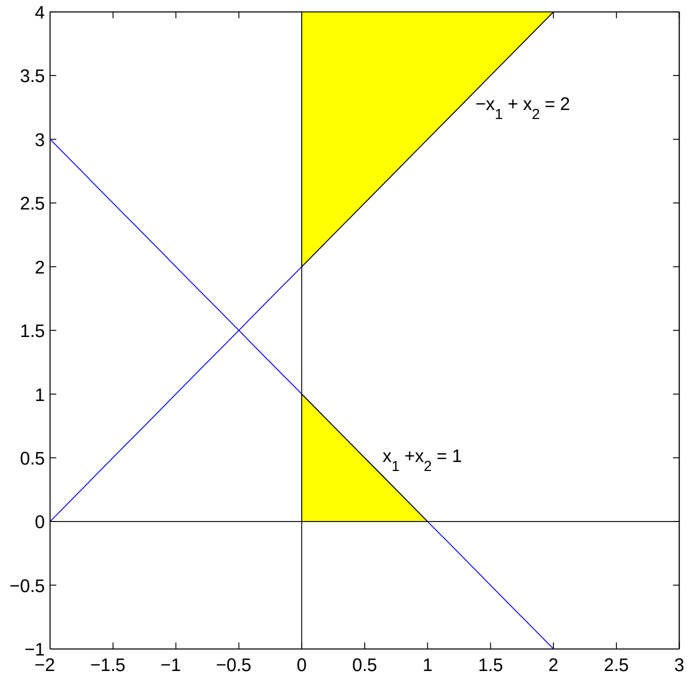

Figure 4.1. *No Feasible Region*

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	-2	0	-1	1	-1	1
x_2	1	1	0	0	1	1
x_0	$-1+2M$	0	M	0	$1+M$	$1-M$

Since M is a very large number, $-1+2M$ is positive. Hence all entries in the x_0 row are nonnegative. Thus we have reached an optimal point. However, we see that the artificial variable $x_4 = 1$, which is not zero. That means that the solution just found is not a solution to our original problem. Indeed the $\mathbf{x}$ that satisfies $A\mathbf{x} + I\mathbf{x}_a = \mathbf{b}$ with $\mathbf{x}_a \neq \mathbf{0}$ is not a solution to $A\mathbf{x} = \mathbf{b}$. Figure 4.1 shows that the feasible region to the problem is empty.

4.2 Unbounded Solutions

Theorem 4.1. *Consider an LPP in feasible canonical form. If in the simplex tableau, there exists a nonbasic variable x_j such that $y_{ij} \leq 0$ for all $i = 1, 2, \cdots, m$, i.e. all entries in the x_j column are non positive, then the feasible region is unbounded. If moreover that $z_j - c_j < 0$, then there exists a feasible solution with at most $m+1$ variables nonzero and the corresponding value of the objective function can be set arbitrarily large.*

Proof. Let $\mathbf{x}_B$ be the current BFS with $B\mathbf{x}_B = \mathbf{b}$. Let the columns of B be denoted by $\mathbf{b}_i$. Then we have

$$B\mathbf{x}_B = \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i = \mathbf{b}.$$

Let $\mathbf{a}_j$ be the column of A that corresponds to the variable x_j. By (**??**), we have

$$\mathbf{a}_j = B\mathbf{y}_j = \sum_{i=1}^{m} y_{ij}\mathbf{b}_i.$$

Hence for all $\theta > 0$, we have

$$\begin{aligned}
\mathbf{b} &= \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i - \theta\mathbf{a}_j + \theta\mathbf{a}_j \\
&= \sum_{i=1}^{m} x_{B_i}\mathbf{b}_i - \theta\sum_{i=1}^{m} y_{ij}\mathbf{b}_i + \theta\mathbf{a}_j \\
&= \sum_{i=1}^{m}(x_{B_i} - \theta y_{ij})\mathbf{b}_i + \theta\mathbf{a}_j \ .
\end{aligned}$$

Thus we obtain a new nonbasic solution of $m+1$ nonzero variables. This solution is feasible as

$$x_{B_i} - \theta y_{ij} \geq 0, \quad \text{for all } i.$$

Moreover, the value of x_j, which is equal to θ, can be set arbitrarily large, indicating that the feasible region is unbounded in the x_j direction.

If moreover that $c_j > z_j$, then the value of the objective function can be set arbitrarily large since

$$\begin{aligned}
\hat{z} &= \sum_{i=1}^{m}(x_{B_i} - \theta y_{ij})c_{B_i} + \theta c_j = \sum_{i=1}^{m} x_{B_i}c_{B_i} - \theta\sum_{i=1}^{m} y_{ij}c_{B_i} + \theta c_j \\
&= \mathbf{c}_B\mathbf{x}_B - \theta\mathbf{c}_B^T\mathbf{y}_j + \theta c_j = z - \theta z_j + \theta c_j = z + \theta(c_j - z_j).
\end{aligned}$$

This proves our assertion. □

Example 4.2. This is an example where the feasible region and the optimal value of the objective function are unbounded. Consider the LPP

$$\begin{array}{ll}
\max & x_0 = 2x_1 + x_2 \\
\text{subject to} & \begin{cases} x_1 - x_2 \leq 10 \\ 2x_1 - x_2 \leq 40 \\ x_1, x_2 \geq 0 \end{cases}
\end{array}$$

The initial tableau is

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_3	1	-1	1	0	10
x_4	2	-1	0	1	40
x_0	-2	-1	0	0	0

No positive ratio exists in x_2 column. Hence x_2 can be increased without bound while maintaining feasibility. It is evident from Figure 4.2.

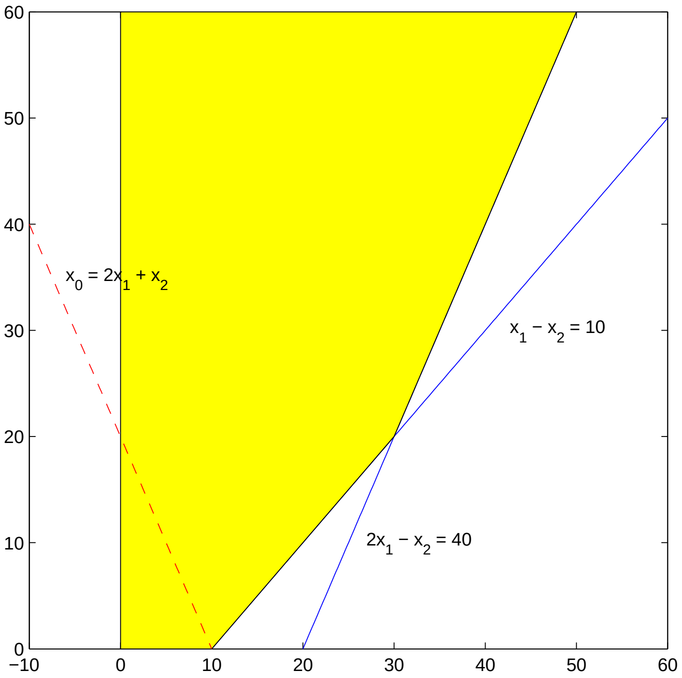

Figure 4.2. *Unbound Feasible Region with Bounded Optimal Value*

Example 4.3. The following is an example where the feasible region is unbounded yet the optimal value is bounded. Consider the LPP

$$\begin{array}{ll} \max & x_0 = 6x_1 - 2x_2 \\ \text{subject to} & \begin{cases} 2x_1 - \ x_2 \leq 2 \\ \ x_1 \qquad \leq 4 \\ \ x_1, \quad x_2 \geq 0 \end{cases} \end{array}$$

The computation goes as follows

	x_1	x_2	x_3	x_4	**b**
x_3	2^*	-1	1	0	2
x_4	1	0	0	1	4
x_0	-6	2	0	0	0

$$\downarrow$$

	x_1	x_2	x_3	x_4	**b**
x_1	1	$-1/2$	$1/2$	0	1
x_4	0	$1/2^*$	$-1/2$	1	3
x_0	0	-1	3	0	6

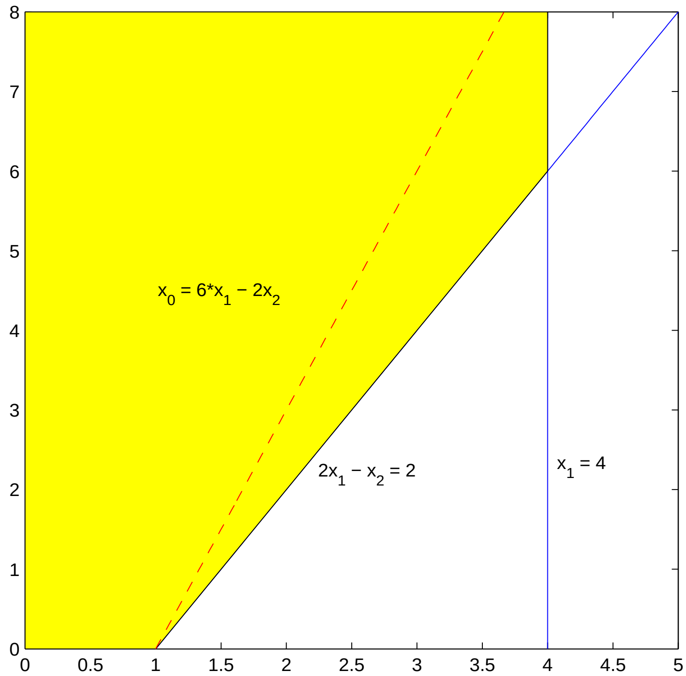

Figure 4.3. *Unbounded Feasible Region with Unbounded Optimal Value*
Any $(x_1, x_2) = (1, k)$ for k being any positive number is a feasible solution.

$\downarrow$

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_1	1	0	0	1	4
x_2	0	1	-1	2	6
x_0	0	0	2	2	12

Optimal tableau

4.3 Infinite Number of Optimal Solutions

Zero reduced cost coefficients for non-basic variables at optimality indicate alternative optimal solutions, since if we pivot in those columns, x_0 value remains the same after a change of basis for a different BFS, see Section 3.5. Notice that simplex method yields only the extreme point optimal (BFS) solutions. More generally, the set of alternative optimal solutions is given by the convex combination of optimal extreme point solutions. Suppose $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^p$ are extreme point optimal solutions, then $\mathbf{x} = \sum_{k=1}^{p} \lambda_k \mathbf{x}^k$, where $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{p} \lambda_k = 1$ is also an optimal solution. In fact, if $\mathbf{c}^T\mathbf{x}^k = z_0$ for $1 \le k \le p$, then

$$\mathbf{c}^T\mathbf{x} = \sum_{k=1}^{p} \lambda_k \mathbf{c}^T\mathbf{x}^k = \sum_{k=1}^{p} \lambda_k z_0 = z_0.$$

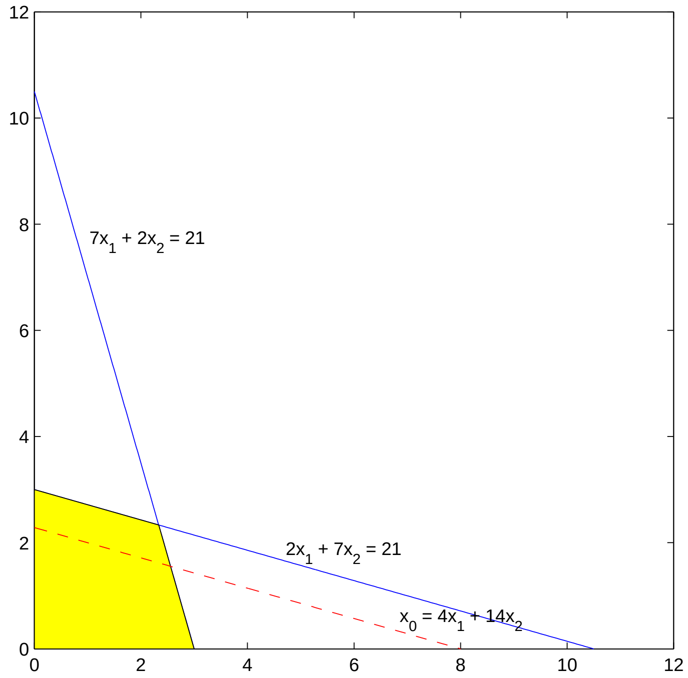

Figure 4.4. *Infinitely many Optimal Solutions*

Any $(x_1, x_2) = (1, k)$ for k being any positive number is a feasible solution.

Example 4.4. Consider

$$\begin{array}{ll} \max & x_0 = 4x_1 + 14x_2 \\ \text{subject to} & \begin{cases} 2x_1 + 7x_2 \le 21 \\ 7x_1 + 2x_2 \le 21 \\ \quad x_1, x_2 \ge 0 \end{cases} \end{array}$$

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_3	2	7^*	1	0	21
x_4	7	2	0	1	21
x_0	-4	-14	0	0	0

$$\downarrow$$

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_2	$2/7$	1	$1/7$	0	3
x_4	$45/7^*$	0	$-2/7$	1	15
x_0	$\boxed{0}$	0	2	0	42

$$\downarrow\uparrow$$

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_2	0	1	7/45	−2/45	7/3
x_1	1	0	−2/45	$7/45^*$	7/3
x_0	0	0	2	$\boxed{0}$	42

Thus all convex combinations of the points $[0, 3, 0, 15]$ and $[7/3, 7/3, 0, 0]$ are optimal feasible solutions.

4.4 Degeneracy and Cycling

Degenerate basic solutions are basic solutions with one or more basic variables at zero level. Degeneracy occurs when one or more of the constraints are redundant.

Example 4.5. Consider the following LLP

$$\begin{array}{ll} \max & x_0 = 2x_1 + x_2 \\ \text{subject to} & \begin{cases} 4x_1 + 3x_2 \le 12 \\ 4x_1 + x_2 \le 8 \\ 4x_1 - x_2 \le 8 \\ \quad x_1,\ x_2 \ge 0 \end{cases} \end{array}$$

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_3	4	3	1	0	0	12
x_4	4^{**}	1	0	1	0	8
x_5	4^*	−1	0	0	1	8
x_0	−2	−1	0	0	0	0

$(*) \swarrow \qquad\qquad \searrow (**)$

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_3	0	4	1	0	−1	4
x_4	0	2^*	0	1	−1	0
x_1	1	−1/4	0	0	1/4	2
x_0	0	−3/2	0	0	1/2	4

Degenerate Vertex { $x_4 = 0$ and basic }

$$\downarrow$$

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_3	0	2^{**}	1	−1	0	4
x_1	1	1/4	0	1/4	0	2
x_5	0	−2	0	−1	1	0
x_0	0	−1/2	0	1/2	0	4

Degenerate Vertex { $x_5 = 0$ and basic }

$$\downarrow$$

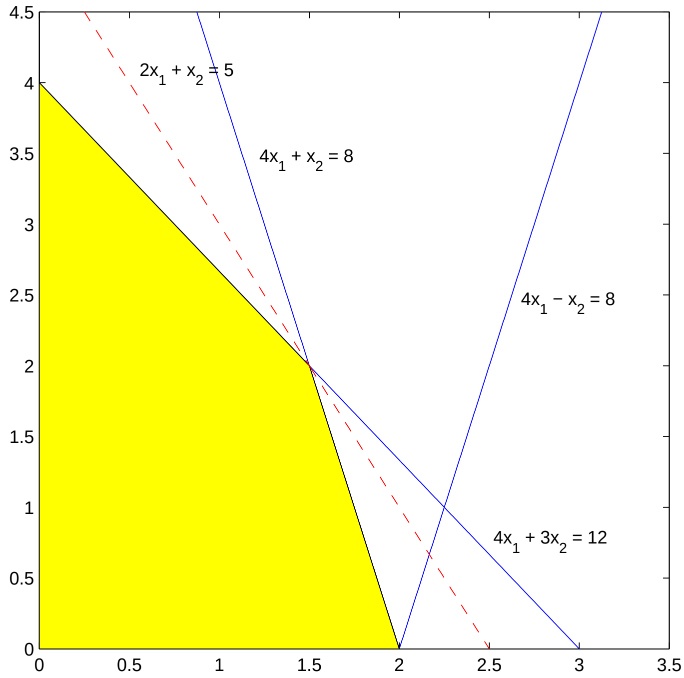

Figure 4.5. *Degeneracy and Cycling*

	x_1	x_2	x_3	x_4	x_5	**b**
x_3	0	0	1	-2	1^*	4
x_2	0	1	0	$1/2$	$-1/2$	0
x_1	1	0	0	$1/8$	$1/8$	2
x_0	0	0	0	$3/4$	$-1/4$	4

	x_1	x_2	x_3	x_4	x_5	**b**
x_2	0	1	$1/2$	$-1/2$	0	2
x_1	1	0	$-1/8$	$3/8$	0	$3/2$
x_5	0	0	1	-2	1	4
x_0	0	0	$1/4$	$1/4$	0	5

Degenerate Vertex { $x_2 = 0$ and basic }

$\downarrow$

	x_1	x_2	x_3	x_4	x_5	**b**
x_5	0	0	1	-2	1	4
x_2	0	1	$1/2$	$-1/2$	0	2
x_1	1	0	$-1/8$	$3/8$	0	$3/2$
x_0	0	0	$1/4$	$1/4$	0	5

In figure 4.5, we see that the degenerate vertex V can be represented either by

$$\{x_2 = 0, x_4 = 0\}, \quad \{x_4 = 0, x_5 = 0\} \quad \text{or} \quad \{x_2 = 0, x_5 = 0\}.$$

We note that degeneracy guarantees the existence of more than one feasible pivot element, i.e. *tie-ratios* exist. For example, in the first tableau, the ratios for variables x_4 and x_5 are both equal to 2.

When an LP is degenerate, i.e. its feasible region possesses degenerate vertices, *cycling* may occur as follows: Suppose the current basis is B and such that this basis B yields a degenerate BFS. Since moving from a degenerate vertex (BFS) to another degenerate vertex does not affect (i.e. increase or decrease) the objective function value. It is then possible for the simplex procedure to start from the current (degenerate) basis B, and after some p iterations, to return to B with no change in the objective function value – as long as all vertices in-between are degenerate. This means that a further p iterations will again bring us back to this same basis B. The process is then said to be cycling.

Example 4.6. The following is an example where cycling occurs.

$$\begin{array}{ll} \max & x_0 = 20x_1 + \frac{1}{2}x_2 - 6x_3 + \frac{3}{4}x_4 \\ \text{subject to} & \begin{cases} x_1 \le 2 \\ 8x_1 - x_2 + 9x_3 + \frac{1}{4}x_4 \le 11 \\ 12x_1 - \frac{1}{2}x_2 + 3x_3 + \frac{1}{2}x_4 \le 24 \\ x_2 \le 1 \\ x_1,\ x_2,\ x_3,\ x_4 \ge 0 \end{cases} \end{array}$$

(T0)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_5	1^*	0	0	0	1	0	0	0	2
x_6	8	-1	9	$1/4$	0	1	0	0	16
x_7	12	$-1/2$	3	$1/2$	0	0	1	0	24
x_8	0	1	0	0	0	0	0	1	1
x_0	-20	$-1/2$	6	$-3/4$	0	0	0	0	0

(T1)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	0	0	1	0	0	0	2
x_6	0	-1	9	$1/4^*$	-8	1	0	0	0
x_7	0	$-1/2$	3	$1/2$	-12	0	1	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	$-1/2$	6	$-3/4$	20	0	0	0	40

(T2)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	0	0	1	0	0	0	2
x_4	0	-4	36	1	-32	4	0	0	0
x_7	0	$3/2$	-15	0	4^*	-2	1	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	$-7/2$	33	0	-4	3	0	0	40

(T3)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	$-3/8$	$15/4$	0	0	$1/2$	$-1/4$	0	2
x_4	0	8^*	-84	1	0	-12	8	0	0
x_5	0	$3/8$	$-15/4$	0	1	$-1/2$	$1/4$	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	-2	18	0	0	1	1	0	40

(T4)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	$-3/16$	$3/64$	0	$-1/16$	$1/8$	0	2
x_2	0	1	$-21/2$	$1/8$	0	$-3/2$	1	0	0
x_5	0	0	$3/16^*$	$-3/64$	1	$1/16$	$-1/8$	0	0
x_8	0	0	$21/2$	$-1/8$	0	$3/2$	-1	1	1
x_0	0	0	-3	$1/4$	0	-2	3	0	40

(T5)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	0	0	1	0	0	0	2
x_2	0	1	0	$-5/2$	56	2^*	-6	0	0
x_3	0	0	1	$-1/4$	$16/3$	$1/3$	$-2/3$	0	0
x_8	0	0	0	$5/2$	-56	-2	6	1	1
x_0	0	0	0	$-1/2$	16	-1	1	0	40

(T6)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	0	0	1	0	0	0	2
x_6	0	$1/2$	0	$-5/4$	28	1	-3	0	0
x_3	0	$-1/6$	1	$1/6$	-4	0	$1/3^*$	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	$1/2$	0	$-7/4$	44	0	-2	0	40

(T7)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_1	1	0	0	0	1	0	0	0	2
x_6	0	-1	9	$1/4^*$	-8	1	0	0	0
x_7	0	$-1/2$	3	$1/2$	-12	0	1	0	0
x_8	0	1	0	0	0	0	0	1	1
x_0	0	$-1/2$	6	$-3/4$	20	0	0	0	40

Note that Tableau 1 is the same as Tableau 7. Thus starting from the basis $B = \{x_1, x_6, x_7, x_8\}$, we have moved to $\{x_1, x_4, x_7, x_8\}$, to $\{x_1, x_4, x_5, x_8\}$, to $\{x_1, x_2, x_5, x_8\}$, to $\{x_1, x_2, x_3, x_8\}$ to $\{x_1, x_6, x_3, x_8\}$

and finally back to $\{x_1, x_6, x_7, x_8\}$ in six iterations, or a cycle of period $p = 6$. To break the cycle, bring in x_4 and remove x_7. Then the next iteration yields the optimal solution $\mathbf{x}^* = [2, 1, 0, 1, 0, 3/4, 0, 0]$ with $x_0^* = 41.25$.

To get out of cycling, one way is to try a different pivot element. This is done as indicated in our example above. Another way in terms of computer implementation is by perturbation of data. For our example, this may be done by changing $\mathbf{b} = [2, 16, 24, 1]^T$ in Tableau 0 to $[2.00001, 16.000001, 24.0000001, 1.00000001]^T$. The little trick will always work as long as the perturbation is sufficiently small. The reason roughly speaking is that non-degenerate BFS are dense in the set of BFS.

4.5 Artificial Variables in Phase II of the Two-phase Method

When applying the two-phase method, if one or more artificial variables appears in the basis at zero level when Phase I ends, then we have a degeneracy in the solution. In Phase II, we must make sure that these artificial variables will never become positive again. Suppose a non-artificial variable x_j is to be inserted into the basis and that $x_i = 0$ is an artificial variable in the basis at zero level.

(1) If $y_{ij} > 0$, then

$$\frac{x_i}{y_{ij}} = 0 = \min\Big\{\frac{x_k}{y_{kj}} : \ y_{kj} > 0\Big\}.$$

Hence x_j enters the basis and replace x_i. Hence no difficulty arises in this case.

(2) If $y_{ij} = 0$, then x_i will not be removed. Let x_r be the leaving variable. Since in the next iteration,

$$\hat{x}_i = x_i - \frac{y_{ij}}{y_{rj}} x_r,$$

we see that $\hat{x}_i$ remains at the zero level. Hence no difficulty arises in this case either.

(3) If $y_{ij} < 0$, then x_i will not be removed. Let x_r be the leaving variable. Since in the next iteration,

$$\hat{x}_i = x_i - \frac{y_{ij}}{y_{rj}} x_r,$$

Hence if $x_r > 0$, then $\hat{x}_i > 0$ and we are in trouble. To bypass this, we must remove the artificial variable x_i instead. Since $y_{ij} \neq 0$, it can be used as a pivot. Thus we are sure that the new solution will still be basic. Moreover, this new solution will still be feasible because the artificial variable was at zero level. In fact in the new solution,

$$\hat{x}_l = x_l - \frac{y_{lj}}{y_{ij}} x_i = x_l, \quad l \neq i,$$

and the new basic variable $\hat{x}_j = x_i / y_{ij} = 0$. Thus we see that the solution is basically unchanged – the new basic variable x_j enter at zero level to replace the leaving variable x_i at zero level. In particular, we have $\hat{z} = z$.

Example 4.7. Consider the following LP.

$$\begin{array}{ll} \max & x_0 = x_1 + 1.5x_2 + 5x_3 + 2x_4 \\ \text{subject to} & \begin{cases} 3x_1 + 2x_2 + \ \ x_3 + 4x_4 \leq 6 \\ 2x_1 + \ \ x_2 + 5x_3 + \ \ x_4 \leq 4 \\ 2x_1 + 6x_2 - 4x_3 + 8x_4 = 0 \\ \ \ x_1 + 3x_2 - 2x_3 + 4x_4 = 0 \end{cases} \end{array}$$

Two artificial variables are added to the last two equations. Notice that the last equation is redundant. Because of this redundancy, we know that one artificial variable will be in the basis at the termination of Phase I.

PHASE I: TABLEAU 0.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\mathbf{b}$
x_5	3	2	1	4	1	0	0	0	6
x_6	2	1	5	1	0	1	0	0	4
x_7	2	6	-4	8	0	0	1	0	0
x_8	1	3	-2	4	0	0	0	1	0
x_0	0	0	0	0	0	0	-1	-1	0

Eliminating the -1 in the x_0 row, we get

PHASE I: TABLEAU 1.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\mathbf{b}$
x_5	3	2	1	4	1	0	0	0	6
x_6	2	1	5	1	0	1	0	0	4
x_7	2	6	-4	8	0	0	1	0	0
x_8	1	3	-2	4	0	0	0	1	0
x_0	3	9	-6	-3	0	0	0	0	0

Since $x_0 = 0$, Phase I ends and the two artificial variables are in the basis at zero level. Assigning the original cost coefficients to the structural variables, we have the initial tableau for the Phase II.

PHASE II: TABLEAU 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\mathbf{b}$
x_5	3	2	1	4	1	0	0	0	6
x_6	2	1	5	1	0	1	0	0	4
x_7	2	6	-4^*	8	0	0	1	0	0
x_8	1	3	-2	4	0	0	0	1	0
x_0	-1	$-\frac{3}{2}$	-5	-2	0	0	0	0	0

If we choose x_3 as the entering variable, then according to the usual feasibility condition, x_6 will be the leaving variable. However, if this were done, both artificial variables would be positive at the next iteration. Hence we must use our alternative rule and remove one of the artificial variables instead. It was arbitrarily decided to remove x_7. Notice that since x_7 is an artificial variable, the column corresponding to x_7 can be removed once x_7 leaves the basis. In the next tableau, we keep it to illustrate a point.

PHASE II: TABLEAU 2.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\mathbf{b}$
x_5	$\frac{7}{2}$	$\frac{7}{2}$	0	6	1	0	–	0	6
x_6	$\frac{9}{2}$	$\frac{17}{2}$	0	11^*	0	1	–	0	4
x_3	$-\frac{1}{2}$	$-\frac{3}{2}$	1	-2	0	0	–	0	0
x_8	0	0	0	0	0	0	$-\frac{1}{2}$	1	0
x_0	$-\frac{7}{2}$	-9	0	-12	0	0	–	0	0

Notice that except for the 1, the x_8 row contains only zeros. Thus we know that the equation corresponds to x_8 is redundant. In fact, if we compute the entries in the x_7 column, we see that the last entry is a $-1/2$, indicating that $-x_7/2 + x_8 = 0$, i.e. $x_7 = 2x_8$. Therefore, we cross off the x_8 row and proceed with a tableau containing one row less than the preceding one.

PHASE II: TABLEAU 3.

	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}$
x_5	$-\frac{23}{22}$	$-\frac{25}{22}$	0	0	1	$-\frac{6}{11}$	$\frac{42}{11}$
x_4	$\frac{9}{22}$	$\frac{17}{22}$	0	1	0	$\frac{1}{11}$	$\frac{4}{11}$
x_3	$\frac{7}{22}$	$\frac{1}{22}$	1	0	0	$\frac{2}{11}$	$\frac{8}{11}$
x_0	$\frac{31}{22}$	$\frac{3}{11}$	0	0	0	$\frac{12}{11}$	$\frac{48}{11}$

The tableau gives the optimal solution. We note that if the last equation had been dropped at the start, we would have obtained the same tableau.