Chapter 5

DUALITY

5.1 The Dual Problems

Every linear programming problem has associated with it another linear programming problem and that the two problems have such a close relationship that whenever one problem is solved, the other is solved as well. The original LPP is called the *primal* problem and the associated LPP is called the *dual* problem. Together they are called a "*dual pair*" (primal + dual) in the sense that the dual of the dual will again be the primal.

Example 5.1. (The Diet Problem) How can a dietician design the most economical diet that satisfies the basic daily nutritional requirements for a good health? For simplicity, we assume that there are only two foods F_1 and F_2 and the daily nutrition required are N_1, N_2 and N_3. The unit cost of the foods and their nutrition values together with the daily requirement of each nutrition are given in the following table.

	F_1	F_2	Daily Requirement
Cost	120	180	–
N_1	1	1	10
N_2	2	4	24
N_3	3	6	32

Let x_j, $j = 1, 2$ be the number of units of F_j that one should eat in order to minimize the cost and yet fulfill the daily nutrition requirement. Thus the problem is to select the x_j such that

$$\min \qquad x_0 = 120x_1 + 180x_2$$

subject to the nutritional constraints:

$$\begin{cases} x_1 + x_2 \geq 10 \\ 2x_1 + 4x_2 \geq 24 \\ 3x_1 + 6x_2 \geq 32 \end{cases}$$

and the non-negativity constraints: $x_j \geq 0$, $j = 1, 2$. In matrix form, we have

$$\min \qquad x_0 = \mathbf{c}^T\mathbf{x}$$

$$\text{subject to} \begin{cases} A\mathbf{x} \geq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

where

$$\mathbf{c} = \begin{bmatrix} 120 \\ 180 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 10 \\ 24 \\ 32 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

Now let us look at the same problem from a pharmaceutical company's point of view. How can a pharmaceutical company determine the price for each unit of nutrient pill so as to maximize revenue, if a synthetic diet made up of nutrient pills of various pure nutrients is adopted? Thus we have three types of nutrient pills P_1, P_2 and P_3. We assume that each unit of P_i contains one unit of the N_i. Let u_i be the unit price of P_i, the problem is to maximize the total revenue u_0 from selling such a synthetic diet, i.e.

$$\max \qquad u_0 = 10u_1 + 24u_2 + 32u_3$$

subject to the constraints that the cost of a unit of synthetic food j made up of P_i is no greater than the unit market price of F_j:

$$\begin{cases} u_1 + 2u_2 + 3u_3 \le 120 \\ u_1 + 4u_2 + 6u_3 \le 180 \\ \qquad u_1, u_2, u_3 \ge 0 \end{cases}$$

In matrix form, the problem is:

$$\begin{array}{ll} \max & u_0 = \mathbf{b}^T\mathbf{u} \\ \text{subject to} & \begin{cases} A^T\mathbf{u} \le \mathbf{c} \\ \mathbf{u} \ge \mathbf{0} \end{cases} \end{array}$$

We said the two problems form a dual pair of linear programming problem, and we will see that the solution to one should lead to the solution of the other.

Definition 5.1. Let $\mathbf{x}$ and $\mathbf{c}$ be column n-vectors, $\mathbf{b}$ and $\mathbf{u}$ be column m-vectors and A be an m-by-n matrix. The *primal* and the *dual* problems can be defined as follows:

Primal	Dual
$\max\ \mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$ $\mathbf{x} \ge \mathbf{0}$	$\min\ \mathbf{b}^T\mathbf{u}$ subject to $A^T\mathbf{u} \ge \mathbf{c}$ $\mathbf{u} \ge \mathbf{0}$

Calling one primal and the other one dual is completely arbitrary for we have the following theorem.

Theorem 5.1. *The dual of the dual is the primal.*

Proof. Transforming the dual into canonical form, we have

$$\begin{array}{ll} \max & u_0' = -\mathbf{b}^T\mathbf{u} \\ \text{subject to} & \begin{cases} -A^T\mathbf{u} \le -\mathbf{c} \\ \mathbf{u} \ge \mathbf{0} \end{cases} \end{array}$$

Taking the dual of this problem, we have

$$\begin{array}{ll} \min & x_0' = -\mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} -A\mathbf{x} \geq -\mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \end{array}$$

which is the same as the primal problem. □

To obtain the dual of an LP problem in standard form:

$$\begin{array}{ll} \max & x_0 = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \end{array}$$

we can first change it into canonical form:

$$\begin{array}{ll} \max & x_0 = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} \leq \mathbf{b} \\ -A\mathbf{x} \leq -\mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \end{array}$$

Then its dual is given by

$$\begin{array}{ll} \min & u_0 = \mathbf{b}^T\mathbf{u}_1 - \mathbf{b}^T\mathbf{u}_2 \\ \text{subject to} & \begin{cases} A^T\mathbf{u}_1 - A^T\mathbf{u}_2 \geq \mathbf{c} \\ \mathbf{u}_1, \mathbf{u}_2 \geq \mathbf{0} \end{cases} \end{array}$$

Letting $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, we finally have

$$\begin{array}{ll} \min & u_0 = \mathbf{b}^T\mathbf{u} \\ \text{subject to} & \begin{cases} A^T\mathbf{u} \geq \mathbf{c} \\ \mathbf{u} \text{ free} \end{cases} \end{array}$$

The following is a general rule of the relationship between a dual pair.

max	$\sum_{j=1}^{n} c_j x_j$	min	$\sum_{i=1}^{m} u_i b_i$
subject to	$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \ (i = 1, 2, \cdots, k)$	subject to	$u_i \geq 0 \ (i = 1, 2, \cdots, k)$
	$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad (i = k+1, \cdots, m)$		u_i free $\quad (i = k+1, \cdots, m)$
	$x_j \geq 0 \quad (j = 1, 2, \cdots, \ell)$		$\sum_{i=1}^{m} u_j a_{ij} \geq c_j \quad (j = 1, 2, \cdots, \ell)$
	x_j free $\quad (j = \ell+1, \cdots, n)$		$\sum_{i=1}^{m} u_i a_{ij} = c_j \quad (j = \ell+1, \cdots, n)$

We observe from the above the following correspondence:

	Primal	Dual
	Max Program	Min Program
c_j:	n obj. ftn. coeff.	n r.h.s.
b_i:	m r.h.s.	m obj. ftn. coeff.
u_i:	k $(\leq)$ constraints	k non-neg. variables
u_i:	$m-k$ $(=)$constraints	$m-k$ free variables
x_j:	ℓ non-neg. variables	ℓ $(\geq)$ constraints
x_j:	$n-\ell$ free variable	$n-\ell$ $(=)$constraints

Example 5.2. Let the original (primal) problem be given by

$$\begin{array}{ll} \max & x_1 + 4x_2 + 3x_3 \\ \text{subject to} & \begin{cases} 2x_1 + 2x_2 + \ \ x_3 \leq 4 \\ \ \ x_1 + 2x_2 + 2x_3 \leq 6 \\ \ \ x_1, \quad x_2, \quad x_3 \geq 0 \end{cases} \end{array}$$

Before transforming it to the dual problem, we first have to standardize the primal problem.

$$\begin{array}{ll} \max & x_1 + 4x_2 + 3x_3 + 0x_4 + 0x_5 \\ \text{subject to} & \begin{cases} 2x_1 + 2x_2 + \ \ x_3 + \ \ x_4 \qquad = 4 \\ \ \ x_1 + 2x_2 + 2x_3 + \qquad + \ \ x_5 = 6 \\ \ \ x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \geq 0 \end{cases} \end{array}$$

The dual of a standardize primal is easily obtained by transposing the equations. For primals that are maximization problems, the duals are minimization problems with $\geq$ signs. All dual variables are assumed to be unrestricted first.

$$\begin{array}{ll} \min & 4u_1 + 6u_2 \\ \text{subject to} & \begin{cases} 2u_1 + \ \ u_2 \geq 1 \\ 2u_1 + 2u_2 \geq 4 \\ \ \ u_1 + 2u_2 \geq 3 \\ \ \ u_1 \qquad \geq 0 \\ \qquad \ \ u_2 \geq 0 \\ \quad u_1, \ u_2 \quad \text{free} \end{cases} \end{array}$$

After simplification, we get our final dual problem.

$$\begin{array}{ll} \min & 4u_1 + 6u_2 \\ \text{subject to} & \begin{cases} 2u_1 + \ \ u_2 \geq 1 \\ 2u_1 + 2u_2 \geq 4 \\ \ \ u_1 + 2u_2 \geq 3 \\ \quad u_1, \ u_2 \geq 0 \end{cases} \end{array}$$

Example 5.3. Let us consider a primal given by

$$\begin{array}{ll} \min & 5x_1 + 6x_2 \\ \text{subject to} & \begin{cases} x_1 + 2x_2 = 5 \\ -x_1 + 5x_2 \geq 3 \\ 4x_1 + 7x_2 \leq 8 \\ x_1 \text{ free}, \ x_2 \geq 0 \end{cases} \end{array}$$

The standardized primal is given by

$$\begin{array}{ll} \min & 5x_1' - 5x_1'' + 6x_2 + 0x_3 + 0x_4 \\ \text{subject to} & \begin{cases} x_1' - x_1'' + 2x_2 = 5 \\ -x_1' + x_1'' + 5x_2 - x_3 = 3 \\ 4x_1' - 4x_1'' + 7x_2 + x_4 = 8 \\ x_1', \ x_1'', \ x_2, \ x_3, \ x_4 \geq 0 \end{cases} \end{array}$$

Since the primal is a minimization problem, the dual problem will be a maximization problem with $\leq$ signs. The dual variables are assumed to be free first.

$$\begin{array}{ll} \max & 5u_1 + 3u_2 + 8u_3 \\ \text{subject to} & \begin{cases} u_1 - u_2 + 4u_3 \leq 5 \\ -u_1 + u_2 - 4u_3 \leq -5 \\ 2u_1 + 5u_2 + 7u_3 \leq 6 \\ -u_2 \leq 0 \\ u_3 \leq 0 \\ u_1, \ u_2, \ u_3 \text{ free} \end{cases} \end{array}$$

The first two inequality constraints combine together to give an equality constraint.

$$\begin{array}{ll} \max & 5u_1 + 3u_2 + 8u_3 \\ \text{subject to} & \begin{cases} u_1 - u_2 + 4u_3 = 5 \\ 2u_1 + 5u_2 + 7u_3 \leq 6 \\ u_2 \geq 0 \\ u_3 \leq 0 \\ u_1, \ u_2, \ u_3 \text{ free} \end{cases} \end{array}$$

By replacing the free variable u_1 by $u_1' - u_1''$ and the negative variable u_3 by $-u_3$, we finally arrive at

$$\begin{array}{ll} \max & 5u_1' - 5u_1'' + 3u_2 - 8u_3 \\ \text{subject to} & \begin{cases} u_1' - u_1'' - u_2 - 4u_3 = 5 \\ 2u_1' - 2u_1'' + 5u_2 - 7u_3 \leq 6 \\ u_1', \ u_1'', \ u_2, \ u_3 \geq 0 \end{cases} \end{array}$$

which is the dual problem of the original primal problem.

Example 5.4. (Transportation Problem) Suppose that there are m sources that can provide materials to n destinations that require the materials. The following is called the *costs and requirements table* for the transportation problem.

$$
\begin{array}{c|cccc|c}
 & & \text{Destination} & & & \text{Supply} \\
\cline{2-5}
 & c_{11} & c_{12} & \cdots & c_{1n} & s_1 \\
\text{Origin} & c_{21} & c_{22} & \cdots & c_{2n} & s_2 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & c_{m1} & c_{m2} & \cdots & c_{mn} & s_m \\
\cline{2-5}
\text{Demand} & d_1 & d_2 & \cdots & d_n &
\end{array}
$$

where c_{ij} is the unit transportation cost from origin i to destination j, s_i is the supply available from origin i and d_j is the demand required for destination j. We assume that total supply equals to total demand, i.e.

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

The problem is to decide the amount x_{ij} to be shipped from i to j so as to minimize the total transportation cost while meeting all demands. That is

$$
\begin{array}{ll}
\min & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to} & \begin{cases} \sum_{j=1}^{n} x_{ij} = s_i & (i = 1, 2, \cdots, m) \\ \sum_{i=1}^{m} x_{ij} = d_j & (j = 1, 2, \cdots, n) \\ x_{ij} \geq 0 & (i = 1, 2, \cdots, m \; ; \quad j = 1, 2, \cdots, m) \end{cases}
\end{array}
$$

The dual is then given by:

$$
\begin{array}{ll}
\max & \displaystyle\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \\
\text{subject to} & \begin{cases} u_i + v_j \quad \leq c_{ij} & (i = 1, 2, \cdots, m \; ; \quad j = 1, 2, \cdots, n) \\ u_i \;, v_j \qquad \text{free} \end{cases}
\end{array}
$$

5.2 Duality Theorems

We first give the relationship between the objective values of the primal and of the dual.

Theorem 5.2 (Weak Duality Theorem). *If* $\mathbf{x}$ *is a feasible solution (not necessarily basic) to the primal and* $\mathbf{u}$ *is a feasible solution (not necessarily basic) to the dual, then*

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{u} \;.$$

Proof. Since $\mathbf{x}$ is a feasible solution to the primal P, we have $A\mathbf{x} \leq \mathbf{b}$. As $\mathbf{u} \geq \mathbf{0}$, we have

$$\mathbf{u}^T A\mathbf{x} \leq \mathbf{u}^T\mathbf{b} = \mathbf{b}^T\mathbf{u}. \tag{5.1}$$

Similarly, since $\mathbf{A}^T\mathbf{u} \geq \mathbf{c}$ and $\mathbf{x} \geq \mathbf{0}$, we have

$$\mathbf{x}^T A^T\mathbf{u} \geq \mathbf{x}^T\mathbf{c}.$$

Taking the transpose and combining with (5.1), we get $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{u}$. $\square$

As an immediate corollary, we have

Theorem 5.3. *If* $\mathbf{x}_0$ *and* $\mathbf{u}_0$ *are feasible solutions to the primal and the dual respectively and if*

$$\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{u}_0,$$

then $\mathbf{x}_0$ *and* $\mathbf{u}_0$ *are optimal solutions to the primal and the dual respectively.*

Proof. For all feasible solutions $\mathbf{x}$ to the primal, by Theorem 5.2, we have

$$\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{u}_0 = \mathbf{c}^T\mathbf{x}_0.$$

Thus $\mathbf{x}_0$ is an optimal solution to primal. Similarly, if $\mathbf{u}$ is any feasible solution to the dual, then

$$\mathbf{b}^T\mathbf{u} \ge \mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{u}_0$$

Thus $\mathbf{u}_0$ is an optimal solution to the dual. □

The converse of this theorem is the strong duality theorem.

Theorem 5.4 (The Strong Duality Theorem). *A feasible solution* $\mathbf{x}_0$ *to the primal is optimal if and only if there exists a feasible solution* $\mathbf{u}_0$ *to the dual such that*

$$\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{u}_0\ . \tag{5.2}$$

In particular, $\mathbf{u}_0$ *is an optimal solution to the dual.*

Proof. The "if" part is Theorem 5.3. Let us now prove the "only if" part. Let the primal be

$$\begin{array}{ll} \max & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} \le \mathbf{b} \\ \ \ \mathbf{x} \ge \mathbf{0} \end{cases} \end{array}$$

Standardizing it, we have

$$\begin{array}{ll} \max & z = \mathbf{c}^T\mathbf{x} + \mathbf{c}_s^T\mathbf{x}_s \\ \text{subject to} & \begin{cases} A\mathbf{x} + \mathbf{x}_s = \mathbf{b} \\ \ \ \mathbf{x},\ \mathbf{x}_s \ge \mathbf{0}\ , \end{cases} \end{array}$$

where $\mathbf{x}_s$ are all slack variables and $\mathbf{c}_s = \mathbf{0}$. Suppose that $\mathbf{x}_0$ is an optimal solution to the problem with basis matrix B. Then

$$z_j \ge c_j, \qquad j = 1, 2, \cdots, n, n+1, \cdots, n+s\ .$$

Since for $1 \le j \le n$, $z_j = \mathbf{c}_B^T\mathbf{y}_j = \mathbf{c}_B^T(B^{-1}\mathbf{a}_j)$. We have, in vector form,

$$A^T(B^{-1})^T\mathbf{c}_B \ge \mathbf{c}\ .$$

This shows that $\mathbf{u}_0 \equiv (B^{-1})^T\mathbf{c}_B$ is a solution to $A^T\mathbf{u} \ge \mathbf{c}$. It remains to show that $\mathbf{u}_0 \ge \mathbf{0}$ and satisfies (5.2).

We first prove that $\mathbf{u}_0 \ge \mathbf{0}$. Since

$$z_{n+j} \ge c_{n+j}, \quad j = 1, 2, \cdots, s,$$

we have

$$\mathbf{a}_{n+j}^T(B^{-1})^T\mathbf{c}_B \ge c_{n+j}, \quad j = 1, 2, \cdots, s.$$

Since x_{n+j} are slack variables, the corresponding columns in A are just the jth unit vector $\mathbf{e}_j$ and the corresponding cost coefficients $c_{n+j} = 0$. Thus

$$\mathbf{e}_j^T (B^{-1})^T \mathbf{c}_B \geq c_{n+j} = 0, \quad j = 1, 2, \cdots, s.$$

Hence $\mathbf{u}_0 = (B^{-1})^T \mathbf{c}_B \geq \mathbf{0}$. Finally, since

$$\mathbf{b}^T \mathbf{u}_0 = \mathbf{b}^T (B^{-1})^T \mathbf{c}_B = \mathbf{c}_B^T B^{-1} \mathbf{b} = \mathbf{c}^T \mathbf{x}_0,$$

we see that $\mathbf{u}_0$ satisfies (5.2) and by Theorem 5.3, it is an optimal solution to the dual problem. □

This theorem also give us an explicit form of the optimal solution to the dual problem

Theorem 5.5. *If B is the basis matrix for the primal corresponding to an optimal solution and $\mathbf{c}_B$ contains the prices of the variables in the basis, then an optimal solution to the dual is given by $(B^{-1})^T \mathbf{c}_B$, i.e., the entries in the x_0 row under the columns corresponding to the slack variables give the values of the dual structural variables. Moreover, the entries in the x_0 row under the columns for the structural variables will give the optimal values of the dual surplus variables.*

Proof. We only have to prove that $\mathbf{u}_B \equiv (B^{-1})^T \mathbf{c}_B$ is given by the entries in the x_0 row under the columns corresponding to the slack variables. In fact, for the slack variables, we have

$$z_{n+j} - c_{n+j} = z_{n+j} = \mathbf{e}_j^T (B^{-1})^T \mathbf{c}_B = \mathbf{u}_{B_j}.$$

Next we prove that the entries in the x_0 row under the columns for the structural variables give the optimal values of the dual surplus variables. Since

$$z_j = \mathbf{c}_B^T \mathbf{y}_j = \mathbf{c}_B^T (B^{-1} \mathbf{a}_j) = \mathbf{a}_j^T B^{-T} \mathbf{c}_B = \mathbf{a}_j^T \mathbf{u}_B, \quad j = 1, \cdots n,$$

and $A^T \mathbf{u}_B - \mathbf{u}_{B_s} = \mathbf{c}$ where $\mathbf{u}_{B_s}$ is the vector of dual surplus variables, we have

$$\mathbf{z} - \mathbf{c} = A^T \mathbf{u}_B - \mathbf{c} = \mathbf{u}_{B_s}.$$

□

Example 5.5. Let the primal problem be

$$\begin{array}{ll} \max & x_0 = 4x_1 + 3x_2 \\ \text{subject to} & \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 6 \\ 8 \\ 7 \\ 15 \\ 1 \end{bmatrix} \\ x_1, x_2 \geq 0. \end{cases} \end{array}$$

Standardizing the problem, we have

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 7 \\ 15 \\ 1 \end{bmatrix}$$

The optimal tableau is given by

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{b}$
x_3	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
x_2	0	1	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	3
x_4	0	0	0	1	$\frac{3}{2}$	$\frac{1}{2}$	0	5
x_1	1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	4
x_7	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	4
x_0	0	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	0	25

Thus the optimal solution is $[x_1, x_2] = [4, 3]$ with $[x_3, x_4, x_5, x_6, x_7] = [2, 5, 0, 0, 4]$. From the x_0 row, we see that the optimal solution to the dual is given by

$$[u_1, u_2, u_3, u_4, u_5, u_6, u_7] = \left[0, 0, \frac{5}{2}, \frac{1}{2}, 0, 0, 0\right].$$

Let us verify this by considering the dual. The dual of the primal is given by

$$\begin{array}{ll} \min & u_0 = 6u_1 + 8u_2 + 7u_3 + 15u_4 + u_5 \\ \text{subject to} & \begin{cases} \begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ u_i \geq 0, i = 1, 2, 3, 4, 5 \end{cases} \end{array}$$

Changing the minimization problem to a maximization problem and using simplex method (or the dual simplex method to be introduced in §4), we obtain the optimal tableau for the dual:

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\mathbf{c}$
u_4	$\frac{1}{6}$	$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
u_3	$-\frac{1}{2}$	$\frac{3}{2}$	1	0	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{5}{2}$
u_0	2	5	0	0	4	4	3	-25

Thus the optimal solution for the dual is $[u_1, u_2, u_3, u_4, u_5] = \left[0, 0, \frac{5}{2}, \frac{1}{2}, 0\right]$ with optimal surplus variables $[u_6, u_7] = [0, 0]$. Notice that the optimal solution to the primal is given by the reduced cost coefficients for u_4 and u_5, i.e. $[x_1, x_2] = [4, 3]$ and the optimal values of the primal slack variables are given by $[x_3, x_4, x_5, x_6, x_7] = [2, 5, 0, 0, 4]$.

5.3 The Existence Theorem and The Complementary Slackness

Theorem 5.6 (Existence Theorem). *(i) An LPP has a finite optimal solution if and only if both it and its dual have feasible solutions.*

(ii) If the primal has an unbounded maximum, then the dual has no feasible solution.

(iii) If the dual has no feasible solution but the primal has, then the primal has an unbounded maximum.

Proof. (i) This follows by Theorem 5.4.
(ii) Proof by contradiction using Theorem 5.2.
(iii) This follows from Theorem 5.4. □

We summarize the results in the following table.

	Primal is feasible	Primal is not feasible
Dual is feasible	both optimal solutions exist	Dual has unbounded solutions
Dual is not feasible	Primal has unbounded solutions	possible

Theorem 5.7 (Complementary Slackness). *Given any pair of optimal solutions to an LP problem and its dual, then*

(i) for each i, $i = 1, 2, \cdots, m$, the product of the ith primal slack variable and ith dual variable is zero, and

(ii) for each j, $j = 1, 2, \cdots, n$, the product of the jth primal variable and jth surplus dual variable is zero.

Proof. Let the primal problem be

$$\begin{array}{ll} \max & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} \le \mathbf{b} \\ \mathbf{x} \ge \mathbf{0} \end{cases} \end{array}$$

Standardizing it, we have

$$\begin{array}{ll} \max & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} + \mathbf{x}_s = \mathbf{b} \\ \mathbf{x},\ \mathbf{x}_s \ge \mathbf{0}\ , \end{cases} \end{array} \tag{5.3}$$

where $\mathbf{x}_s$ are the slack variables. Standardizing the dual, we have

$$\begin{array}{ll} \max & u_0 = \mathbf{b}^T\mathbf{u} \\ \text{subject to} & \begin{cases} A^T\mathbf{u} - \mathbf{u}_s = \mathbf{c} \\ \mathbf{u}, \mathbf{u}_s \ge \mathbf{0} \end{cases} \end{array} \tag{5.4}$$

where $\mathbf{u}_s$ are the surplus variables. Given any feasible primal solution $[\mathbf{x}, \mathbf{x}_s]$ and any column m-vectors $\mathbf{u}$, (5.3) implies that

$$\mathbf{u}^T A\mathbf{x} + \mathbf{u}^T\mathbf{x}_s = \mathbf{u}^T\mathbf{b} = \mathbf{b}^T\mathbf{u}. \tag{5.5}$$

Given any feasible dual solution $[\mathbf{u}, \mathbf{u}_s]$ and any column n-vector $\mathbf{x}$, (5.4) implies that

$$\mathbf{x}^T A^T \mathbf{u} - \mathbf{x}^T \mathbf{u}_s = \mathbf{x}^T \mathbf{c} = \mathbf{c}^T \mathbf{x}, \tag{5.6}$$

Since the cost coefficients of all slack and surplus variables are zero, we see that if $[\mathbf{x}_0, \mathbf{x}_{0s})]$ and $[\mathbf{u}_0, \mathbf{u}_{0s})]$ are optimal solutions to the primal and the dual problems, then

$$\mathbf{c}^T \mathbf{x}_0 = \mathbf{b}^T \mathbf{u}_0.$$

Hence by (5.5) and (5.6), we have

$$\mathbf{u}_0^T \mathbf{x}_{0s} + \mathbf{x}_0^T \mathbf{u}_{0s} = 0.$$

Using the fact that $\mathbf{u}_0$, $\mathbf{x}_{0s}$, $\mathbf{x}_0$, $\mathbf{u}_{0s} \geq \mathbf{0}$, we finally have $\mathbf{u}_0^T \mathbf{x}_{0s} = 0 = \mathbf{x}_0^T \mathbf{u}_{0s}$. □

We remark that the converse of Theorem 5.7 is also true.

Theorem 5.8. *Let $[\mathbf{x}_0, \mathbf{x}_{0s}]$ be feasible solution to primal and $[\mathbf{u}_0, \mathbf{u}_{0s}]$ be feasible solution to dual. Suppose that*

(i) for each i, $i = 1, 2, \cdots, m$, the product of the ith primal slack variable and ith dual variable is zero, and

(ii) for each j, $j = 1, 2, \cdots, n$, the product of the jth primal variable and jth surplus dual variable is zero.

Then $[\mathbf{x}_0,\, \mathbf{x}_{0s}]$ and $[bou_0, \mathbf{u}_{0s}]$ are optimal solutions to the primal and the dual respectively.

Proof. By assumption, $\mathbf{u}_0^T \mathbf{x}_{0s} + \mathbf{x}_0^T \mathbf{u}_{0s} = \mathbf{0}$. Hence

$$\mathbf{u}_0^T \mathbf{x}_{0s} = -\mathbf{x}_0^T \mathbf{u}_{0s} = -\mathbf{u}_{0s}^T \mathbf{x}_0.$$

Adding the term $\mathbf{u}_0^T A \mathbf{x}_0$ to both sides, we have

$$\mathbf{u}_0^T (A\mathbf{x}_0 + \mathbf{x}_{0s}) = (\mathbf{u}_0^T A - \mathbf{u}_{0s}^T)\mathbf{x}_0.$$

Since $[\mathbf{x}_0,\, \mathbf{x}_{0s}]$ and $[\mathbf{u}_0, \mathbf{u}_{0s}]$ are feasible solutions to the primal and the dual,

$$\mathbf{u}_0^T \mathbf{b} = \mathbf{c}^T \mathbf{x}_0.$$

Thus by Theorem 5.3, both solutions are optimal solutions. □

To see why we have the complementary slackness, suppose that the jth surplus variable of the dual problem is positive. Then by Theorem 5.5, the reduced cost coefficient of the jth structural variable of the primal problem is negative (because it is equal to the negation of the jth surplus variable of the dual problem). Hence the jth primal structural variable should be equal to zero if it is at the optimum. For if not, then we can set it to zero and thus increase the objective value.

Example 5.6. (Dual Prices) Let the primal be given by

$$\begin{array}{rr} \max & x_1 + 4x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 2x_2 + \ \ x_3 \leq 4 \\ & x_1 + 2x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Its dual is

$$\begin{aligned} \min \quad & 4u_1 + 6u_2 \\ \text{subject to} \quad & 2u_1 + \ \ u_2 \geq 1 \\ & 2u_1 + 2u_2 \geq 4 \\ & u_1 + 2u_2 \geq 3 \\ & u_1, u_2 \geq 0 \end{aligned}$$

Initial Tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	2	2	1	1	0	4
x_5	1	2	2	0	1	6
x_0	-1	-4	-3	0	0	0

Optimal Tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_2	$\frac{3}{2}$	1	0	1	$-\frac{1}{2}$	1
x_3	-1	0	1	-1	1	2
x_0	2	0	0	1	1	10

Thus the optimal primal solution is $\mathbf{x}^* = [0, 1, 2, 0, 0]$ and by the duality theorem, the optimal dual solution is $\mathbf{u}^* = [1, 1, 2, 0, 0]$. Let us check for the complementary slackness for these two dual solutions.

$$\begin{array}{lllllll} u_1^* > 0 & \Rightarrow & x_4^* = 0 & \Rightarrow & 2x_1^* + 2x_2^* + x_3^* = 4 & \text{i.e.} & 2(0) + 2(1) + 2 = 4 \\ y_2^* > 0 & \Rightarrow & x_5^* = 0 & \Rightarrow & x_1^* + 2x_2^* + 2x_3^* = 6 & \text{i.e.} & 0 + 2(1) + 2(2) = 6 \\ x_1^* = 0 & \Rightarrow & u_3^* \geq 0 & \Rightarrow & 2u_1^* + u_2^* \geq 1 & \text{i.e.} & 2(1) + 1 = 3 \geq 1 \\ x_2^* > 0 & \Rightarrow & u_4^* = 0 & \Rightarrow & 2u_1^* + 2u_2^* = 4 & \text{i.e.} & 2(1) + 2(1) = 4 \\ x_3^* > 0 & \Rightarrow & u_5^* = 0 & \Rightarrow & u_1^* + 2u_2^* = 3 & \text{i.e.} & 1 + 2(1) = 3 \end{array}$$

5.4 Dual Simplex Method

In the usual simplex method, which will be called *primal method* for distinction, we start with a primal BFS $\mathbf{x}$, maintain primal feasibility $\{x_{i0} \geq 0\}_{i=1}^m$ and strive for non-positivity of the reduced cost coefficients (which is equivalent to $\{x_{0j} \geq 0\}_{j=1}^n$). However, by Theorem 5.5, the entries in the x_0 row give the values of the dual variables at optimal. Thus the nonnegativity of $\{x_{0j} \geq 0\}_{j=1}^n$ is equivalent to the feasibility of the dual variables.

In the *dual method*, we start with a dual BFS $\mathbf{u}$, maintain dual feasibility $\{u_{j0} \geq 0\}_{j=1}^n$ (which is equivalent to $\{x_{0j} \geq 0\}_{j=1}^n$) and strive for nonnegativity of $\{u_{0i} \geq 0\}_{i=1}^m$ (which is equivalent to primal feasibility $\{x_{i0} \geq 0\}_{i=1}^m$).

Since at any iteration, both the primal and the dual solutions have the same objective value, by the duality theorem, we see that if both solutions are feasible, then we have reached optimality.

Algorithm for the dual simplex method

1. Given a *dual* BFS $\mathbf{x}_B$, if $\mathbf{x}_B \geq \mathbf{0}$, then the current solution is optimal; otherwise select an index r such that the component x_r of $\mathbf{x}_B$ is negative.
2. If $y_{rj} \geq 0$ for all $j = 1, 2, \cdots, n$, then the dual is unbounded; otherwise determine an index s such that
$$-\frac{y_{0s}}{y_{rs}} = \min_j \left\{ -\frac{y_{0j}}{y_{rj}} \middle| y_{rj} < 0 \right\} .$$
3. Pivot at element y_{rs} and return to step 1.

Example 5.7. Consider the problem:

$$\begin{array}{rl} \min & 3x_1 + 4x_2 + 5x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

In canonical form, it is

$$\begin{array}{rl} \max & -3x_1 - 4x_2 - 5x_3 \\ \text{subject to} & -x_1 - 2x_2 - 3x_3 \leq -5 \\ & -2x_1 - 2x_2 - x_3 \leq -6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The initial tableau is

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	-1	-2	-3	1	0	-5
x_5	-2^*	-2	-1	0	1	-6
x_0	3	4	5	0	0	0
ratios	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{1}$	–	–	

After one iteration

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	0	-1^*	$-\frac{5}{2}$	1	$-\frac{1}{2}$	-2
x_1	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	3
x_0	0	1	$\frac{7}{2}$	0	$\frac{3}{2}$	-9
ratios	–	$\frac{1}{1}$	$\frac{7}{5}$	–	$\frac{3}{1}$	

Optimal Tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_2	0	1	$\frac{5}{2}$	-1	$\frac{1}{2}$	2
x_1	1	0	-2	1	-1	1
x_0	0	0	1	1	1	-11

Since both the primal and the dual solutions are feasible, we have reached the optimal solution. The primal optimal solution is given by $\mathbf{x}^* = [1, 2, 0]$, the dual optimal solution is $\mathbf{u}^* = [1, 1]$ and the optimal objective value is 11 for the original problem is a minimization problem.

5.5 Post-Optimality or Sensitivity Analysis

Given an LP problem, suppose that we have found the optimal feasible solution by the simplex (or dual simplex) method. *Post-optimality* or *sensitivity analysis* is the study of how the changes in the original LP problem would affect the feasibility and optimality of the current optimal solution. Before we analyze the method, we first recall the following criteria for determining the optimal primal solutions.
Primal feasibility:

$$A\mathbf{x} = \mathbf{b} \Leftrightarrow B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b}. \tag{5.7}$$

Primal optimality:

$$\mathbf{z}^T - \mathbf{c}^T = \mathbf{c}_B^T B^{-1} A - \mathbf{c}^T \geq \mathbf{0}. \tag{5.8}$$

In the following we will consider changes in the original problem that can affect only one of these criteria. For in these cases, we can obtain the new optimal solution without redoing the whole simplex method for the new LP.

(1) *Changes in resource vector* $\mathbf{b}$.
From (5.7) and (5.8), we see that changes in $\mathbf{b}$ will affect the feasibility but not the optimality of the current optimal solution. Thus if the current optimal solution satisfies the old constraints with the new right hand sides, then it will be the new optimal solution. By the duality theory, the changes in $\mathbf{b}$ will affect the optimality but not the feasibility of the dual optimal solution. In fact, the cost vector for the dual problem is given by $\mathbf{b}$.

(2) *Changes in cost/profit vector* $\mathbf{c}$.
By the duality theory, (or from (5.7) and (5.8) again), we see that changes in the cost vector $\mathbf{c}$ will affect only the optimality of the primal optimal solution and the feasibility of the dual optimal solution. Thus if the current optimal solution satisfies the criteria that the new x_0 row is nonnegative, then it will be the optimal solution for the new LP.

(3) *Changes in technology matrix* A.
If the changes in A occur at the basic variables, then B will be changed. From (5.7) and (5.8), we see that both the feasibility and the optimality of the current optimal solution may be violated. In that case, we have to redo the whole problem. However, if the changes of A are restricted to columns of nonbasic variables (i.e. N in (5.7)), then we see that only dual feasibility (or equivalently primal optimality) will be affected because $\mathbf{x}_N = \mathbf{0}$.

(4) *Addition of a new primal variable/dual constraint* $a_{ij} + c_j$.
This case is essentially the same as considering *simultaneously* changes in the objective function coefficient as well as the corresponding technological coefficients of nonbasic variable. (One can assume that the a_{ij} and the c_j are originally there with values equal to zero.) Consequently, the addition of a new variable can only affect the optimality of the problem. This means that the new variable will enter the solution if, and only if, it improves the objective function value. Otherwise the new variable becomes just another nonbasic variable (= 0).

(5) *Addition of a new primal constraint/dual variable* $a_{ij} + b_i$.
A new constraint can affect the feasibility of the current optimal solution only if it is *active*, i.e. it is not redundant with respect to the current optimal solution. Consequently, the first step would be to check whether the new constraint is satisfied by the current optimal solution. If it is satisfied, the new constraint is redundant and the optimal solution remains unchanged. Otherwise, the new constraint must be added to the system and the dual simplex method is used to clear the primal infeasibility (dual optimality).

Example 5.8. Consider the LPP given by the following tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	1	3	4	1	0	30
x_5	0	4	-1	0	1	10
x_0	-2	-7	3	0	0	0

The optimal tableau is:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	0	-1	5	1	-1	20
x_1	1	4	-1	0	1	10
x_0	0	1	1	0	2	20

(1) *Changes in resource vector* $\mathbf{b}$.
Let the new $\hat{\mathbf{b}} = [10, 20]^T$. Then the new basic solution is given by

$$\hat{\mathbf{x}}_B = B^{-1}\hat{\mathbf{b}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} -10 \\ 20 \end{bmatrix}$$

Thus it is no longer feasible. The new objective value is

$$\hat{x}_0 = \mathbf{c}_B^T \hat{\mathbf{x}}_B = [0, 2] \begin{bmatrix} -10 \\ 20 \end{bmatrix} = 40.$$

We then need to apply the dual simplex method to restore primal feasibility with a new $\mathbf{b}$ column of $[-10, 20, 40]^T$. The new starting tableau is given by:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	0	-1^*	5	1	-1	-10
x_1	1	4	-1	0	1	20
x_0	0	1	1	0	2	40

(2) *Changes in cost/profit vector* $\mathbf{c}$.
Let the new $\hat{\mathbf{c}} = [3, 6, -3, 0, 0]^T$. The new x_0 row is given by

$$\hat{\mathbf{z}}^T - \hat{\mathbf{c}}^T = \hat{\mathbf{c}}_B^T B^{-1} A - \hat{\mathbf{c}}^T = [0, 6, 0, 0, 3] \geq \mathbf{0}.$$

(Recall that $B^{-1}A$ is just the last tableau.) This indicates primal optimality. Thus the primal optimal solution is unchanged. Looking at the dual, the new dual variables are

$$\hat{\mathbf{u}} = B^{-T}\hat{\mathbf{c}}_B = B^{-T}\begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}.$$

(3) *Changes in technology coefficients* a_{ij}.
In the optimal tableau, x_1 and x_4 are basic. Thus we can only change the entries of $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_5$. Let the new $\hat{\mathbf{a}}_2 = [1, 5]^T$. While the primal feasibility remains, we need to calculate the new reduced cost coefficient for x_2.

$$\hat{z}_2 - c_2 = \mathbf{c}_B^T B^{-1}\hat{\mathbf{a}}_2 - c_2 = [0, 2]\begin{bmatrix}1\\5\end{bmatrix} - 7 = 3 \geq 0\ .$$

Thus the current basis remains optimal with $\mathbf{x}_B = (10, 20)^T$ and $x_0 = 20$ unchanged. However, if $\hat{\mathbf{a}}_2 = [1, 3]^T$, then $\hat{z}_2 - c_2 = -1 \leq 0$, indicating non-optimality. In this case, we need to replace the column under x_2 in the (previously optimal) tableau by

$$\hat{\mathbf{y}}_2 = B^{-1}\hat{\mathbf{a}}_2 = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}$$

and pivot in the x_2 column once (to have x_2 become basic) to restore optimality. The new optimal $\mathbf{x}_B = [x_4, x_2]^T = [26\frac{2}{3}, 3\frac{1}{3}]^T$ with $x_0 = 23\frac{1}{3}$.

Example 5.9. (Adding extra constraints) Consider the following LLP problem:

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_3	-1	-1	1	0	-1
x_4	-2	-3	0	1	-2
x_0	3	1	0	0	0

We note that we have primal infeasibility and dual feasibility. Using the dual simplex method, we get the following optimal tableau.

	x_1	x_2	x_3	x_4	$\mathbf{b}$
x_2	1	1	-1	0	1
x_4	1	0	-3	1	1
x_0	2	0	1	0	-1

Since both the primal and the dual are feasible, we have reached the optimal solutions with $\mathbf{x}^* = [0, 1, 0, 1]$ and $\mathbf{u}^* = [1, 0]$. Suppose now a new constraint $x_1 - x_2 \geq 1$ is added. As $x_1^* - x_2^* = 0 - 1 = -1 < 1$, $\mathbf{x}^*$ is infeasible. Adding slack variable x_5 gives $x_5 - x_1 + x_2 = -1$. Let us add this constraint into the optimal tableau.

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_2	1	1	-1	0	0	1
x_4	1	0	-3	1	0	1
x_5	-1	1	0	0	1	-1
x_0	2	0	1	0	0	-1

Notice that for the basic variable x_2, its column is not a unit vector. Thus it is not a simplex tableau. Using pivot operation, we change it back to the unit vector and we have the following simplex tableau.

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_2	1	1	-1	0	0	1
x_4	1	0	-3	1	0	1
x_5	-2^*	0	1	0	1	-2
x_0	2	0	1	0	0	-1

We note that now the primal becomes infeasible again. Using dual simplex method, we finally arrive at the new optimal tableau.

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_2	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0
x_4	0	0	$-\frac{5}{2}$	1	$\frac{1}{2}$	–
x_1	1	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1
x_0	0	0	2	0	1	-3

Thus new $\mathbf{x}^* = [1, 0, 0, 0, 0]$ and new $\mathbf{u}^* = [0, 1]$.

This idea of treating additional constraints can sometimes be exploited to possibly reduce the overall computational effort of solving an LP. Since the computational difficulty depends far more heavily on the number of constraints than the number of variables, it may be possible first to relax the constraints which one suspects are not binding. The new relaxed problem is then solved with a fewer number of constraints. After the optimal solution of the new problem is obtained, the deleted constraints are checked for feasibility in the spirit of such sensitivity analysis. It is common to refer to these constraints as *secondary constraints.*