RESEARCH SUMMARY

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The main focus of my research has been on problems in computational quasi-conformal geometry and their applications. My research interests span the areas of scientific computing, medical imaging, computer visions and computer graphics. The main goal is develop mathematical models and algorithms to effectively study geometric structures, using quasi-conformal Teichmüller theory as a tool. For example, in Human Brain Mapping, neuroscientists commonly aim to identify structural differences between healthy and unhealthy brains, in order to detect systematic patterns of alterations in brain diseases. The main obstacle is that the human brain is a very complicated manifold with difficult geometry. It is extremely difficult for neuroscientists to analyze diseases accurately and efficiently by directly looking at the brain. Developing mathematical models with computational differential geometry to systematically analyze the complicated anatomical structure is crucial for disease analysis. In computer graphics, building mathematical models for the geometry processing of 3D shapes are necessary for 3D visualization. Quasi-conformal geometry has found important applications along this direction.

My research mainly contributes to the following three aspects. Firstly, I am interested in the surface registration/parameterization problems. The goal is to create a meaningful one-to-one pointwise correspondence between two corresponding surfaces that matches geometric information. Surface registration allows us to compare different surfaces systematically, which is especially important in medical shape analysis. Using quasi-conformal Teichmüller theory, my collaborators and I have developed various registration models for surfaces with arbitrary topologies, including the landmark-based, intensity-based and hybrid registration models. These models have found to be effective in computing the surface registration accurately and efficiently. Secondly, I am interested in the scientific computing problems related to computational conformal/quasi-conformal geometry. Conformal/quasi-conformal mappings have been extensively applied in medical imaging, computer graphics and computer visions. Despite their usefulness, developing fast and accurate computational algorithms is definitely necessary to boast up their practicability in real world applications. Motivated by this, we have developed computational algorithms to compute conformal/quasi-conformal parameterizations efficiently and accurately. Our algorithms allow the computation of QC parameterization in real time, while attaining the same numerical accuracies as other state-of-the-art algorithms. When extra constraints are enforced, a mapping that is exactly conformal may not exist. An optimized conformal mapping, called the extremal Teichmuller map (T-Map), that satisfies the extra constraints is theoretically guaranteed. However, despite its existence, an effective computational algorithm to obtain such an extremal T-Map was still lacking. We have recently developed an iterative method, called the Quasi-conformal (QC) iterations, which allows us to compute the T-Map efficiently and accurately. The convergence of the iterative scheme to our desired T-Map is also theoretically proven. Thirdly, mathematical shape analysis has also been one of my research focuses. Mathematical shape analysis models aim to compare geometric shapes quantitatively. Usually, suitable metric has to be defined on the shape space, so that its mathematical structure can be used to analyze the collection of geometric shapes. We have built shape analysis models for both 2D and 3D geometric shapes, using quasi-conformal theories. These models have been used in computer visions and medical imaging applications. In particular, our models have been applied to analyze brain cortical surfaces and their sub-cortical structures (such as hippocampus) for the disease analysis of Alzheimer's disease. Our models help to reveal geometric differences of anatomical structures between normal controls and diseased patients, and also locate abnormalities on organs of disease patients.

Broadly speaking, the goal of my research is to develop computational algorithms for quasi-conformal geometry, understand their theoretical aspects and applies them to real-world applications. Our proposed algorithms have been successfully applied in medical imaging for disease analysis, computer graphics and computer visions. In the following sections, I am going to summarize my selected recent research.

Summary of selected recent research

I. Extremal Teichmüller map (T-Map) and its applications

Conformal mappings have been widely used in computer graphics, computer visions and medical imaging to obtain a diffeomorphism between shapes that minimizes angular distortion. Conformal registrations are beneficial since it preserves the local geometry well. However, when extra constraints (such as landmark constraints) are enforced, conformal mappings generally do not exist. In practical applications, it is often desirable to obtain mapping that matches prescribed feature landmarks consistently while minimizing the local geometric distortion as good as possible. This motivates us to look for an optimal quasi-conformal registration, which satisfies the required constraints while minimizing the conformality distortion. Under suitable condition on the constraints, a unique diffeomporphism, called the Teichmuller mapping (T-Map) between two surfaces can be obtained, which minimizes the maximal conformality distortion. Although the existence of such a mapping is theoretically guaranteed, a numerical algorithm to efficiently construct the T-Map was still lacking.

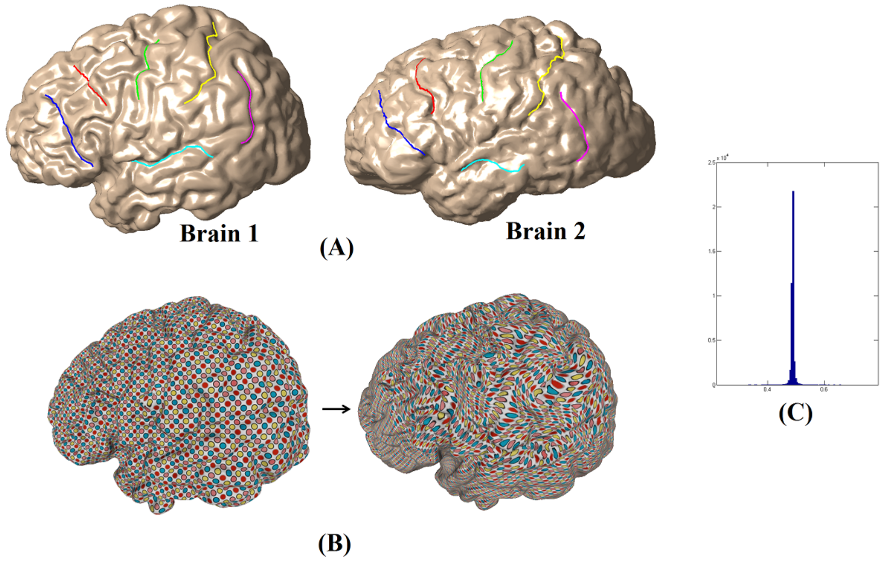

Figure 1: (A) shows 2 brain surfaces with 6 corresponding landmarks. (B) shows the T-Map with 6 landmark constraints enforced. (C) shows the histogram of the norm of BC.

In [1, 2] , we propose an efficient iterative algorithm, called the Quasi-conformal (QC) iterations, to compute the T-Map. The basic idea is to represent the set of diffeomorphisms using Beltrami coefficients (BCs), and look for an optimal BC associated to the desired Teichmuller mapping. The associated diffeomorphism can be efficiently reconstructed from the optimal BC using the Linear Beltrami Solver(LBS). More specifically, the iterative scheme simply consists of the following simple steps:

$$\begin{aligned} f_{n+1} &= \mathbf{LBS}(\mu_{n+1}), \\ \widetilde{\mu}_{n+1} &= \mu_n + \alpha\mu(f_{n+1}, \mu_n), \\ \mu_{n+1} &= \mathcal{L}(\mathcal{P}(\widetilde{\mu}_{n+1})) \end{aligned} \tag{1}$$

where f_n is the quasi-conformal map obtained at the n^{th} iteration, ν_n is the Beltrami differential of f_n and μ_n is a Beltrami differential of constant modulus. $\mathbf{LBS}(\mu)$ is the operator to obtain a quasi-conformal mapping whose Beltrami differential is closest to μ in the least square sense. In other words,

$$\mathbf{LBS}(\mu) = \mathbf{argmin}_{f\in\mathcal{A}}\{\int_{S_1} |\frac{\partial f}{\partial \bar{z}} - \mu\frac{\partial f}{\partial z}|^2 dS_1\} \tag{2}$$

$\mu(f_{n+1}, \nu_n)$ denotes the Beltrami differential of f_{n+1} under the auxiliary metric with respect to ν_n, namely, $|dz+\nu_n d\bar{z}|^2$ ($|dz|^2$ is the original metric on S_1).

Using BCs to represent diffeomorphisms guarantees the diffeomorphic property of the registration. Using the proposed method, the Teichmuller mapping can be accurately and efficiently computed within 10 seconds. The obtained registration is guaranteed to be bijective. The proposed algorithm can be applied on simply-connected surfaces or surfaces of arbitray topologies (such as high-genus surfaces or multiply-connected open surfaces). The convergence of the proposed QC iteration to the extremal T-Map has been theoretically shown in [3].

This proposed algorithm has been practically applied to real applications. In medical imaging, extremal T-Map has been applied to brain landmark matching registration and vertebrae bone registration [1, 4] (see Figure 1). T-Map has also been applied to obtain constrained texture mapping in computer graphics (see Figure 2) and face recognition in computer visions [1].

II. Registration with large deformations via quasi-conformal maps

In some practical situations, images or surfaces may undergo large deformations. In order to compare these corresponding images or surfaces, finding registrations with large deformations is necessary. However, the registration problem is especially challenging when large deformations occur. Foldings or flips in the obtained registration map often exist, which

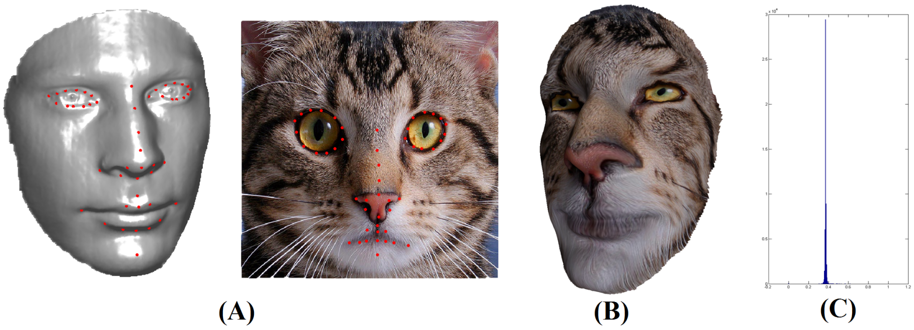

Figure 2: (A) shows a human face and a texture image of a cat. Corresponding landmark points are labeled on the surface and the texture image. We compute the T-Map that matches the landmark points. The T-Map is used as constrained texture mapping to project the texture image onto the surface, as shown in (B). (C) shows the histogram of the norm of BC.

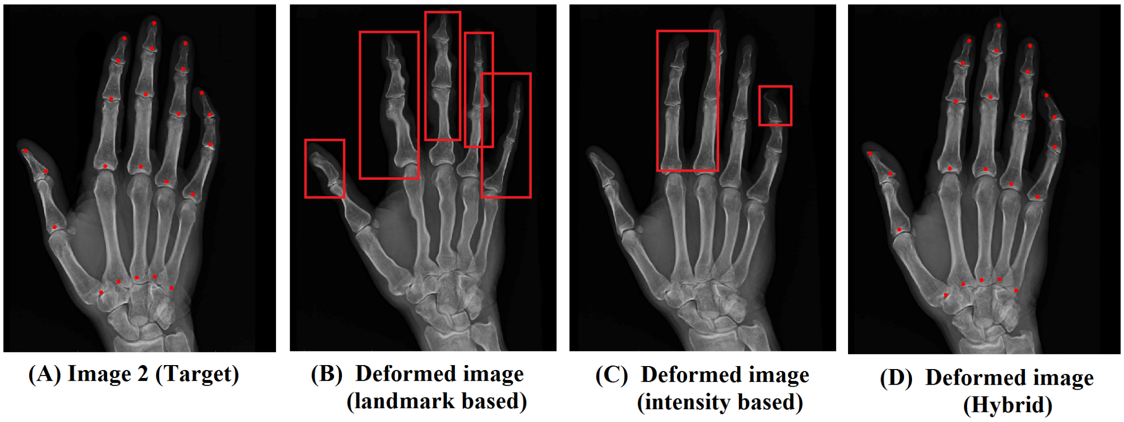

Figure 3: Registration results of the human hand images using different approaches. (A) shows the target image. (B) shows the deformed image from Image 1 using the landmark based registration model. (C) shows the deformed image from Image 1 using the intensity based registration model. (D) shows the deformed image from Image 1 using the proposed hybrid registration model.

are unnatural in many medical imaging or computer visions applications. To solve this problem effectively, we have developed a novel algorithm in [5] to obtain diffeomorphic image or surface registrations with large deformations via quasi-conformal maps. The basic idea is to minimize an energy functional involving a Beltrami coefficient term, which measures the distortion of the quasi-conformal map:

$$f = \mathbf{argmin}_{g:S_1 \to S_2} E_{LM}(\mu(g)) := \mathbf{argmin}_{g:S_1 \to S_2} \left\{ \int_{S_1} |\nabla \mu(g)|^2 + \alpha \int_{S_1} |\mu(g)|^p \right\} \tag{3}$$

subject to:

- $\mathbf{C}$(i) $f(p_i) = q_i$ for $1 \leq i \leq m$; (Landmark constraints) (4)
- $\mathbf{C}$(ii) $||\mu(f)||_\infty < 1$. (bijectivity) (5)

where $\mu(f)$ and $\mu(g)$ are the Beltrami coefficients of f and g respectively.

The Beltrami coefficient effectively controls the bijectivity and smoothness of the registration. Using the proposed algorithm, landmark-matching diffeomorphic (1-1 and onto) registrations between images or surfaces can be effectively obtained, even with a large deformation or large number of feature landmark constraints. The proposed algorithm can also be extended to a hybrid registration model, called Q-Fibra, which combines landmark and intensity (such as image intensity or surface curvature) information to obtain a more accurate registration:

$$f := \mathbf{argmin}_{g:S_1 \to S_2} \left\{ \int_{S_1} |\nabla \mu(g)|^2 + \alpha \int_{S_1} |\mu(g)|^p + \beta \int_{S_1} (I_1 - I_2(g))^2 \right\}. \tag{6}$$

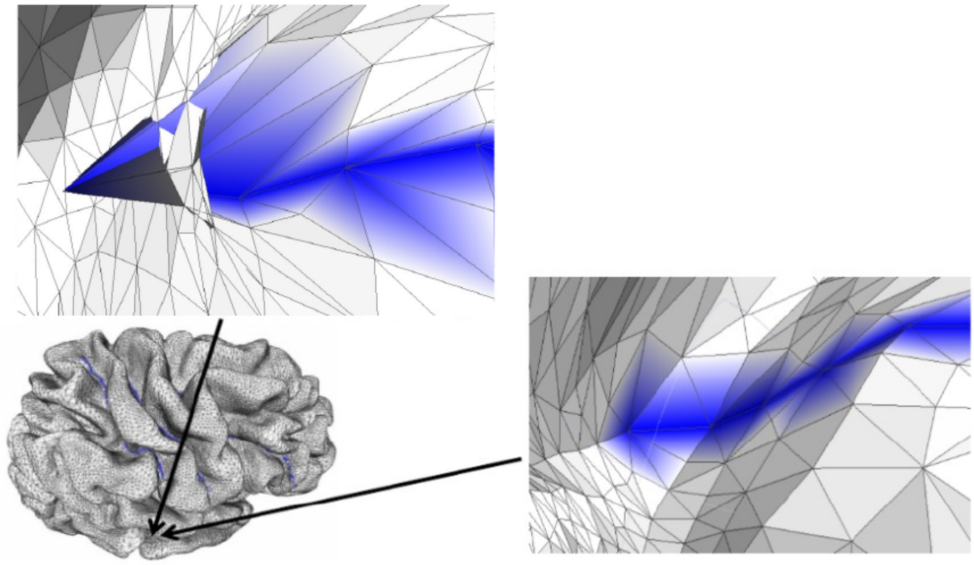

Figure 4: Comparison between the registration result of *FLASH* and that of the conventional algorithm. For the conventional algorithm, foldings are observed near a landmark curve (blue) . On the other hand, *FLASH* preserves the bijectivity of the registration. Top left: The result of the conventional algorithm. Bottom right: The result of *FLASH*.

subject to the constraints C(i) and C(ii).

Experiments have been carried out on both synthetic and real data (see Figure 3). Results has demonstrated the efficacy of the proposed algorithm to obtain diffeomorphic registrations between images or surfaces.

III. Fast global optimized conformal parameterization for Riemann surfaces

Surface registration between cortical surfaces is crucial in medical imaging to perform systematic comparisons between brains. Landmark-matching registration that matches anatomical features, called the sulcal landmarks, are often required, to obtain a meaningful 1-1 correspondence between brain surfaces. This is commonly done by parameterizing the surface onto a simple parameter domain, such as the unit sphere, which aligns sulcal landmarks consistently. Landmark-matching surface registration can then be obtained from the composition map of the parameterizations. For genus-0 closed brain surfaces, optimized spherical harmonic parameterization, which aligns landmarks to consistent locations on the sphere, have been widely used. This approach performs well under small deformations. However, the bijectivity is usually lost when large deformations occur or large amount of landmark constraints are enforced (see Figure 4). Besides, the algorithm involves solving an optimization problem over the space of all diffeomorphisms from the surface onto the sphere, which is nonlinear. Hence, the computation is slow. To alleviate this issue, we have developed in [6] a fast algorithm (called *FLASH*) to compute the optimized landmark aligned spherical harmonic parametreization. This is done by formulating the optimization problem to the extended complex plane $\overline{\mathbb{C}}$ and thereby linearizing the problem. Error introduced near the pole (or the infinity point in $\overline{\mathbb{C}}$) is corrected by composing the mapping with any quasi-conformal map to remove the conformality distortion. More precisely, we look for a quasi-conformal map $\tilde{g}_i : \mathbb{C} \to \mathbb{C}$ such that:

$$\begin{aligned} g|_{\Omega_C} &= \mathbf{id}|_{\Omega_C}; \\ \mu(\tilde{g}_i)|_{\mathbb{C}\setminus\Omega_C} &= \widetilde{\mu}_{\mathbb{C}\setminus\Omega_C} \end{aligned} \tag{7}$$

where $\mu(\tilde{g}_i)$ is the Beltrami differential of g_i. By this treatment, the conformality distortion of $g_i \circ \phi_i$ near the south pole, which is negligible, will remain unchanged. On the other hand, the conformality distortions near the north pole can be much improved

Furthermore, by adjusting the Beltrami differential of the mapping, which measures the conformality distortion, a diffeomorphic (1-1, onto) spherical parameterization can be effectively obtained (see Figure 4). This is done by a simple thresholding procedure of the Beltrami coefficient. When flips occur, the modulus of the Beltrami coefficient ν_0 at that region is greater than 1. We chop down $|\nu^0|$ by:

$$|\widetilde{\nu}(z)| = \min\{|\nu^0(z)|, 1-\epsilon\} \tag{8}$$

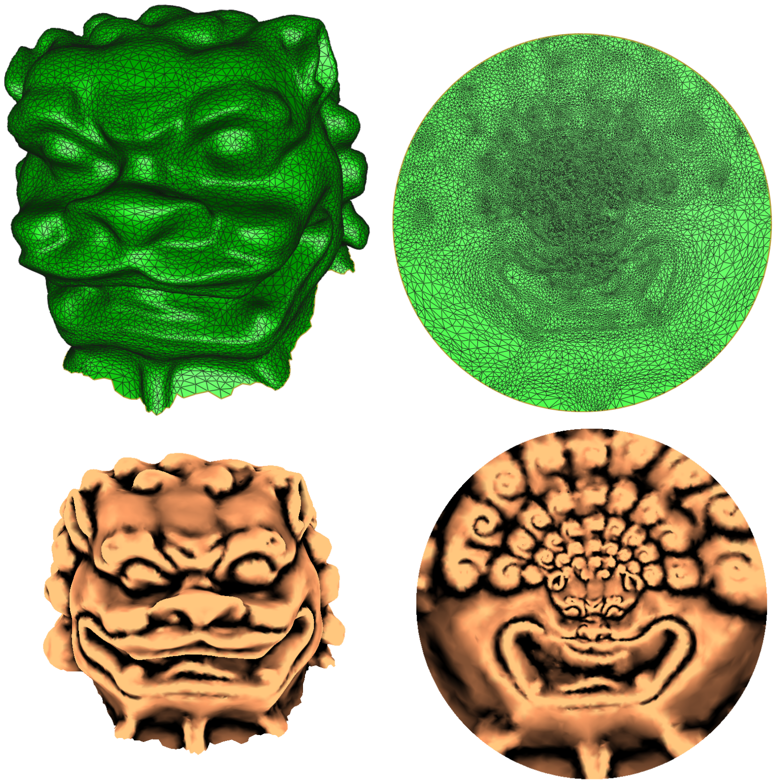

Figure 5: A Chinese lion head and its disk conformal parameterization using our proposed method. Top: the triangulations. Bottom: the mean curvature texture maps.

for all $z \in \mathbb{C} \setminus \Omega$ and $\epsilon > 0$ is a small constant. We then look for a smooth approximation μ_{smooth} of $\widetilde{\nu}$ by:

$$\mu^0_{smooth} = \mathbf{argmin}_{\mu} \int (|\nabla \mu|^2 + |\mu - \widetilde{\nu}|^2 + \frac{1}{D}|\mu|^2) \tag{9}$$

where $D : \mathbb{C} \setminus \Omega \to \mathbb{R}^+$ and $D(p)$ is defined as the geodesic distance between $(0, 0, 1)$ and the image of $p \in \mathbb{C} \setminus \Omega$ under inverse stereographic projection. Here $|\nabla\mu|^2$ measures the smoothness of the Beltrami differential, $|\mu - \widetilde{\nu}|^2$ measures the difference between the original and new Beltrami differential, and D gives a weight for limiting the changes of the Beltrami differential for points close to ∞ (which corresponds to the north pole in $\mathbb{S}^2$). After μ^0_{smooth} is computed, we reconstruct the quasi-conformal map g^0 with the corresponding Beltrami differential μ^0_{smooth} with landmarks fixed. We keep the procedure going until the resulting map $\widetilde{\phi}^n_1$ becomes a diffeomorphism.

Using the proposed algorithm, the computation of the optimized spherical harmonic parameterization with consistent landmark alignment can be significantly speeded up (100 times faster than the conventional method). Experiments have been carried out on real human brain surfaces, which demonstrate the effectiveness of the proposed algorithm.

The proposed algorithm can also be immediately applied to compute disk conformal parameterizations for simply-connected open surfaces, which is also a non-linear problem. Despite the effectiveness of the state-of-the-art approaches, there are still opportunities for further enhancements in the computational time and the conformality of the parameterizations. Firstly, as most of the latest algorithms are nonlinear, the computation is quite inefficient. This becomes an obstacle for practical applications in which a large number of surfaces are involved. Secondly, the conformality distortion is still far from negligible. The distortion affects the accuracy of the parameterizations, and thus hinders practical applications. There are two sources of the conformality distortion. One of the sources is the discretization of the surfaces and the operators in different algorithms. Since the surfaces are usually represented as triangulated meshes, the operators are discretized. Although the conformality of the parameterizations is theoretically guaranteed in the continuous case, certain numerical angular distortions inevitably exist for the discrete case under any algorithms. Another source is the limitations of the algorithms themselves due to different assumptions and conditions in the algorithms. Following the idea of FLASH algorithm, we have developed in [7] a numerical method for disk conformal parameterizations that overcomes

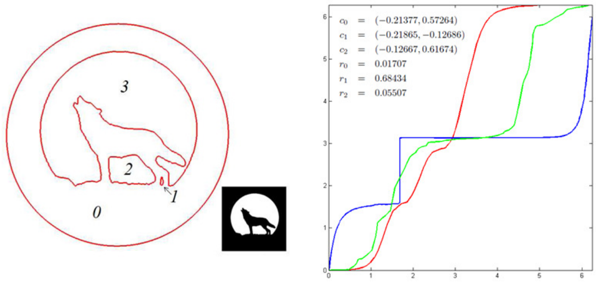

Figure 6: Shape signatures of the wolf image with 3 boundaries and 1 levels.

the above mentioned obstacles. First, we propose to speed up the computation by linearizing the algorithm as much as possible through Cayley transform. To enhance the accuracy, we then propose a simple two-step iteration to correct the conformality distortion with the aid of quasi-conformal theories. Thirdly, our proposed method for disk conformal parameterizations is bijective. The bijectivity is ensured by the property of the Beltrami differential of the composition map. Experimental results suggest that our proposed method outperforms other state-of-the-art approaches in terms of both efficiency and accuracy (see Figure 5).

IV. Shape analysis of planar multiply-connected objects using conformal welding

Shape analysis is a central problem in the field of computer vision. In 2D shape analysis, classification and recognition of objects from their observed silhouettes are extremely crucial but difficult. It usually involves an efficient representation of 2D shape space with a metric, so that its mathematical structure can be used for further analysis. Although the study of 2D simply-connected shapes has been subject to a corpus of literatures, the analysis of *multiply-connected* shapes is comparatively less studied. In [8], we have proposed a representation for general 2D multiply-connected domains with arbitrary topologies using *conformal welding*. A metric can be defined on the proposed representation space, which gives a metric to measure dissimilarities between objects. The main idea is to map the exterior and interior of the domain conformally to unit disks and circle domains (unit disk with several inner disks removed), using holomorphic 1-forms. A set of diffeomorphisms of the unit circle $\mathbb{S}^1$ can be obtained, which together with the conformal modules are used to define the shape signature. More precisely, the signature of a family of non-intersecting planar closed curves $\Gamma = \{\gamma_0, \gamma_1, \cdots, \gamma_n\}$ is defined as

$$S(\Gamma) := \{Mod(D_0), \cdots, Mod(D_s)\} \cup \{f_{ij}\}_{(i,j)\in I},$$

where $Mod(D_j)$'s are the collection of conformal modules and f_{ij}'s are the collection of conformal welding map of each planar closed curve. A shape distance between shape signatures can be defined to measure dissimilarities between shapes (see Figure 6). We have theoretically proved that the proposed shape signature uniquely determines the multiply-connected objects under suitable normalization. We also introduce a reconstruction algorithm to obtain shapes from their signatures. This completes our framework and allows us to move back and forth between shapes and signatures. With that, a morphing algorithm between shapes can be developed through the interpolation of the Beltrami coefficients associated with the signatures. Extensive testing has been carried out on real data and results has demonstrated the efficacy of our proposed algorithm as a stable shape representation scheme.

V. Conformal approaches for geometry processing

In [9], we addressed the problem of surface inpainting, which aims to fill in holes or missing regions on a Riemann surface based on its surface geometry. This is an important problem in geometry processing. In practical situation, surfaces obtained from range scanners often have holes or missing regions where the 3D models are incomplete. In order to analyze the 3D shapes effectively, restoring the incomplete shape by filling in the surface holes is necessary. We propose a novel conformal approach to inpaint surface holes on a Riemann surface based on its surface geometry. The basic idea is to represent the Riemann surface using its conformal factor and mean curvature. According to Riemann surface theory, a Riemann surface can be uniquely determined by its conformal factor and mean curvature up to a rigid motion. Given a Riemann surface S, its mean curvature H and conformal factor λ can be computed easily through its conformal parameterization. Conversely, given λ and H, a Riemann surface can be uniquely reconstructed by solving the Gauss-Codazzi equation on the conformal parameter domain. Hence, the conformal factor and the mean curvature are two geometric quantities fully describing the surface. With this $\lambda - H$ representation of the surface, the problem of surface inpainting can be reduced to the problem of image inpainting of λ and H on the conformal parameter domain. The inpainting of λ and H can be done by conventional image inpainting models. Once λ and H are inpainted, a Riemann

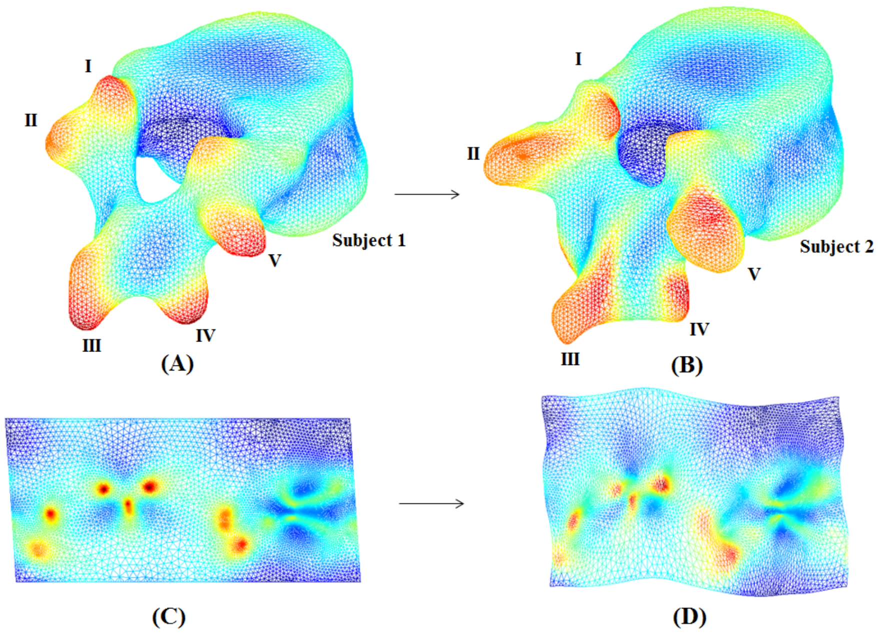

Figure 7: The registration result of the vertebrae bones using our proposed algorithm. (A) shows the vertebrae bone surface of subject 1, colored by its mean curvature. The color intensity (given by the mean curvature) on the vertebrae bone of Subject 1 is mapped to the vertebrae bone of Subject 2 in (B), using the obtained registration. Note that the high curvature regions are consistently matched. (C) and (D) shows the registration result on the universal covering spaces.

surface can be reconstructed which effectively restores the 3D surface with missing holes. Since the inpainting model is based on the geometric quantities λ and H, the restored surface follows the surface geometric pattern as much as possible. We test the proposed algorithm on synthetic data, 3D human face data and MRI-derived brain surfaces. Experimental results show that our proposed method is an effective surface inpainting algorithm to fill in surface holes on an incomplete 3D models based onl their surface geometry.

VI. Geometric registration of high-genus surfaces

In [10], we have presented a method to obtain geometric registrations between high-genus ($g \geq 1$) surfaces. Surface registration between simple surfaces, such as simply-connected open surfaces, has been well studied. However, very few works have been carried out for the registration of high-genus surfaces. The high-genus topology of the surface poses a great challenge for surface registration. A possible approach is to partition surfaces into simply-connected patches and registration can be done in a patch-by-patch manner. Consistent cuts are required, which are usually difficult to obtain and prone to error. We have proposed an effective way to obtain geometric registration between high-genus surfaces without introducing consistent cuts. The key idea is to conformally parameterize the surface into its universal covering space, which is either the Euclidean plane or the hyperbolic disk embedded in $\mathbb{R}^2$. Registration can then be done on the universal covering space by iteratively minimizing a shape mismatching energy measuring the geometric dissimilarity between the two surfaces. More specifically, we proposed to minimize the following energy functional on the universal covering space:

$$E_H(g) = \frac{1}{2}\int_{\tilde{S}_1} |\nabla g|^2 + \frac{\alpha^2}{2}\int_{\tilde{S}_1} (\tilde{H}_1 - \tilde{H}_2 \circ g)^2 + \frac{\beta^2}{2}\int_{\tilde{S}_1} (\tilde{K}_1 - \tilde{K}_2 \circ g)^2 \tag{10}$$

subject to the constraint that $\varphi_i(g^*(a_i)) = g^*(a_i^{-1})$ and $\phi_i(g^*(b_i)) = g^*(b_i^{-1})$ for all $1 \leq i \leq g$. H_i and K_i are the mean curvatures and Gaussian curvatures on S_i respectively. φ_i and ϕ_i are the deck transformations. When $g = 1$, φ_i and ϕ_i are just translations in $\mathbb{R}^2$. When $g > 1$, φ_i and ϕ_i are Mobiüs transformations of the unit disk, which can be computed explicitly. The Beltrami coefficient of the mapping is considered and adjusted in order to control the bijectivity of the mappings in each iterations. Our proposed algorithm effectively computes a smooth registration between high-genus

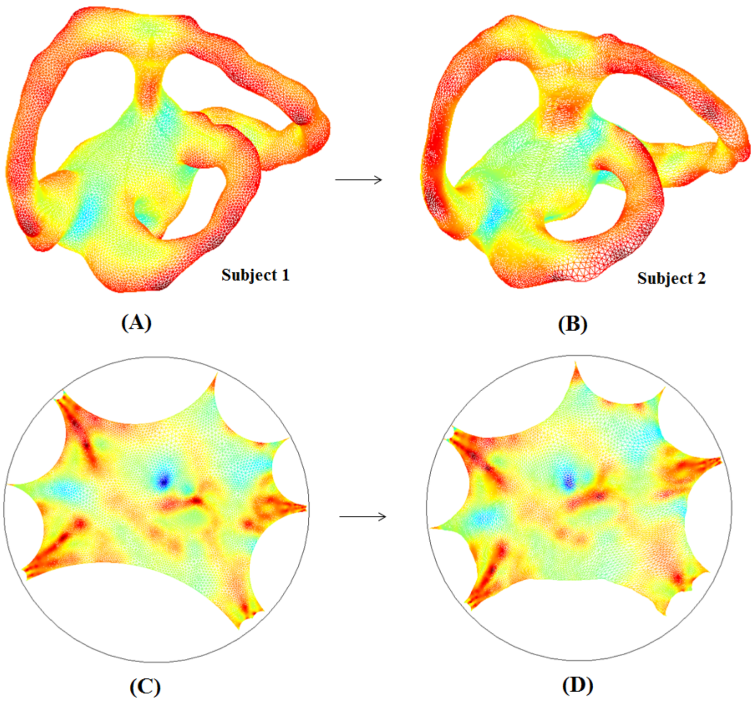

Figure 8: The registration result of the vestibular system using our proposed algorithm. (A) shows the vestibular system surface of subject 1, colored by its mean curvature. The color intensity (given by the mean curvature) on the vestibular system of Subject 1 is mapped to the vestibular system of Subject 2 in (B), using the obtained registration. Note that the corresponding regions are consistently matched. (C) and (D) shows the registration result on the universal covering spaces.

surfaces that matches geometric information as much as possible. The algorithm can also be applied to find a smooth registration minimizing any general energy functionals. Numerical experiments on high-genus surface data show that our proposed method is effective for registering high-genus surfaces with geometric matching. We also applied the method to register anatomical structures, such as the vestibular system and vertebrae bone for medical imaging, which demonstrates the usefulness of the proposed algorithm (see Figure 7 and 8).

VII. Quasi-conformal shape analysis models for medical shape analysis

Our developed quasi-conformal models have been extensively applied to medical imaging and medical shape analysis. Computing surface registration of medical shapes is crucial in medical imaging for systematic comparison between anatomical structures. Anatomical structures are often geometrically and topologically complicated surfaces. This causes the surface registration problem very challenging. Using our quasi-conformal parameterization techniques, we have successfully applied them to the surface registration of vestibular system, brainstem surfaces and vertebrae bone. For medical shape analysis, conformal geometry and quasi-conformal Teichmüller theories have been successfully applied to reveal geometric difference of anatomical structures between normal and diseased subjects. In [11], we have proposed an effective method to analyze shapes of high-genus vestibular system (VS) surfaces by considering their geodesic spectra. The key is to compute the canonical hyperbolic geodesic loops of the surface, using the Ricci flow method. The Fuchsian group generators are then computed which can be used to determine the geodesic spectra. The geodesic spectra effectively measure shape differences between high-genus surfaces up to the hyperbolic isometry. Our method has been applied to real VS data and revealed statistical shape difference in the VS between right-thoracic Adolescent Idiopathic Scoliosis (AIS) patients and normal subjects. For hippocampal surface analysis, we have proposed in [12] a complete shape index using the Beltrami coefficient and curvatures, which measures subtle local differences. The proposed shape energy is zero if and only if two shapes are identical up to a rigid motion. We then seek for the best surface registration by minimizing the shape energy, using a method, called the *Beltrami holomorphic flow* [13, 14]. The complete shape index associated to the optimal surface registration can be used to detect local shape difference between hippocampal surfaces. The method has been applied to real hippocampal surfaces of Alzheimer's disease (AD) patients. Using our proposed algorithm, we successfully located

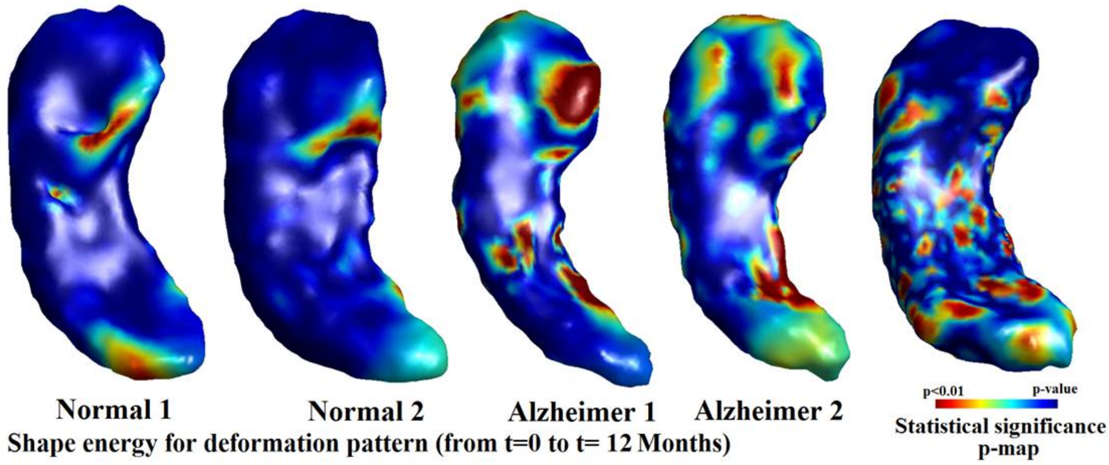

Figure 9: Shape deformation pattern on the hippocampal surfaces of normal controls and Alzheimer's disease patients.

regions of geometric difference with high statistical significance between (AD) and diseased subjects (see Figure 9).

VIII. N-dimensional large deformation registration

Most of the developed quasi-conformal models work on 2-dimensional images or surfaces. In some situations, we may need to compute registration for general n-dimensional spaces. For instance, one may need to compute pointwise correspondence between 3D volumetric brain data. Motivated by this, we are interested to extend our 2D quasi-conformal models to general n-dimensional spaces. In [15], we have proposed a new method to obtain landmark-matching transformations between n-dimensional Euclidean spaces with large deformations. Given a set of feature correspondences, our algorithm searches for an optimal folding-free mapping that satisfies the prescribed landmark constraints. The standard conformality distortion defined for mappings between 2-dimensional spaces is first generalized to the n-dimensional conformality distortion $K(f)$ for a mapping f between n-dimensional Euclidean spaces ($n \geq 3$). More precisely, the conformality distortion $Kf(x)$ of a mapping f at point x is defined by

$$Kf(x) \quad := \quad \begin{cases} \frac{1}{n}\left(\frac{||Df(x)||_F^2}{\det(Df(x))^{2/n}}\right) & \text{if } \det\left(Df(x)\right) > 0, \\ +\infty & \text{otherwise} \end{cases} \tag{11}$$

where $||Df(x)||_F^2 = \mathrm{Tr}(Df(x)^T Df(x))$ denotes the Frobenius norm of $Df(x)$.

We then propose a variational model involving $K(f)$ to tackle the landmark-matching problem in higher dimensional spaces as follows:

$$f^* := \mathbf{argmin}_{f\in F}\|Kf(x)\|_1 + \frac{\sigma}{2}||\Delta f(x)||_2^2 dx \tag{12}$$

where $\sigma \geq 0$ is a fixed parameter and $F = \{f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n | f(p_i) = q_i, i = 1, 2, \ldots m\}$ is the set of functions $f : \Omega \to \mathbb{R}^n$ which satisfies the landmark constraint $f(p_i) = q_i$, where p_i and q_i are the given landmark points ($i = 1, 2, \ldots, m$).

The generalized conformality term $K(f)$ enforces the bijectivity of the optimized mapping and minimizes its local geometric distortions even with large deformations. Another challenge is the high computational cost of the proposed model. To tackle this, we have also proposed a numerical method to solve the optimization problem more efficiently. Alternating direction method with multiplier (ADMM) is applied to split the optimization problem into two subproblems. Preconditioned conjugate gradient method with multi-grid preconditioner is applied to solve one of the sub-problems, while a fixed-point iteration is proposed to solve another subproblem. Experiments have been carried out on both synthetic examples and real data such as lung CT images to compute the diffeomorphic landmark-matching transformation with different landmark constraints. Experimental results has demonstrated the efficacy of our proposed model to obtain a folding-free landmark-matching transformation between n-dimensional spaces even with large deformations (see Figure 10).

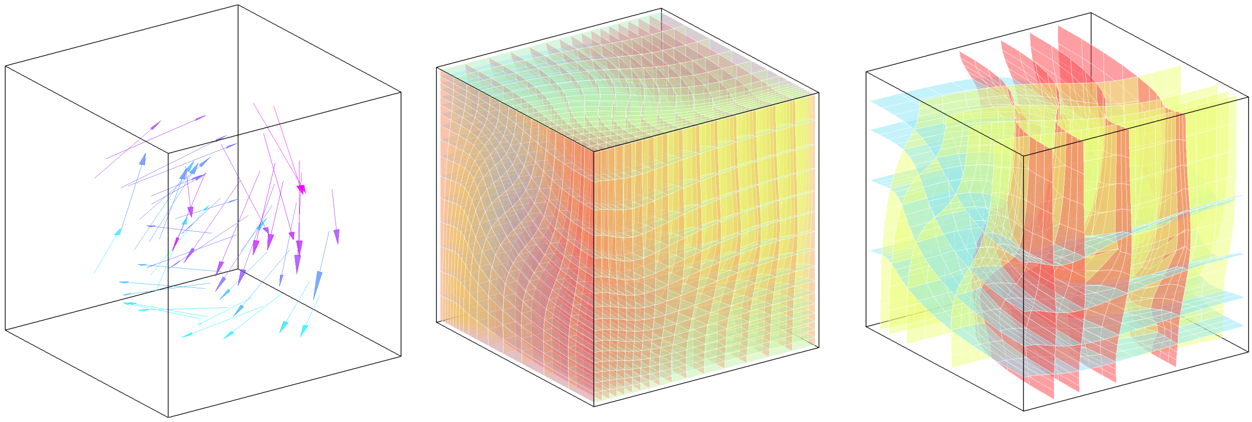

Figure 10: Landmark-matching experiment of random points with twisting deformation.

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